Frobenius-Perron Operator

2015 ACC Workshop on Uncertainty Analysis & Estimation

Raktim Bhattacharya

Laboratory For Uncertainty Quantification
Aerospace Engineering, Texas A&M University.

uq.tamu.edu

Frobenius-Perron Operator

Linear Operator $\mathcal{P}_t$

Given dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(t, \boldsymbol{x}),$$

with $p(t_0, \boldsymbol{x})$ as the initial state density function.

- Evolution of density is given by

$$p(t, \boldsymbol{x}) := \mathcal{P}_t\, p(t_0, \boldsymbol{x}).$$

- $\mathcal{P}_t$ has following properties

$$\mathcal{P}_t\,(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 \mathcal{P}_t\, p_1 + \lambda_2 \mathcal{P}_t\, p_2 \quad \text{linearity}$$

$$\mathcal{P}_t\, p \geq 0 \text{ if } p \geq 0, \quad \text{positivity}$$

$$\int_{\mathcal{X}} \mathcal{P}_t\, p(t_0, \boldsymbol{x}) \mu(d\boldsymbol{x}) = \int_{\mathcal{X}} p(t_0, \boldsymbol{x}) \mu(d\boldsymbol{x}) \quad \text{measure preserving}$$

$\mathcal{P}_t$ is defined by

$$\frac{\partial p}{\partial t} + \boldsymbol{\nabla} \cdot (p\boldsymbol{F}) = 0$$

- Continuity equation
- FPK without diffusion term
- First order linear PDE

First Order PDEs

Method of Characteristics

$$
\begin{aligned}
\frac{\partial p}{\partial t} + \nabla \cdot (p\boldsymbol{F}) &= \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial\, pF_i(t,\boldsymbol{x})}{\partial x_i} \\
&= \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i(t,\boldsymbol{x}) + p \sum_{i=1}^{n} \frac{\partial F_i(t,\boldsymbol{x})}{\partial x_i} = 0
\end{aligned}
$$

This is of the form

$$
a(t,\boldsymbol{x},p)p_t + \sum_i b_i(t,\boldsymbol{x},p)p_{x_i} = c(t,\boldsymbol{x},p).
$$

Lagrange-Charpit equations

$$
\frac{dt}{a(t,\boldsymbol{x},p)} = \frac{dx_i}{b_i(t,\boldsymbol{x},p)} = \frac{dp}{c(t,\boldsymbol{x},p)}
$$

Characteristic Equations

Lagrange-Charpit equations

$$\frac{dt}{a(t,\boldsymbol{x},p)} = \frac{dx_i}{b_i(t,\boldsymbol{x},p)} = \frac{dp}{c(t,\boldsymbol{x},p)}$$

- Let s be parameterization of characteristic curves
- Characteristic curves are given by the ODEs

$$\begin{aligned}\frac{dt}{ds} &= a(t,\boldsymbol{x},p)\\ \frac{dx_i}{ds} &= b_i(t,\boldsymbol{x},p)\\ \frac{dp}{ds} &= c(t,\boldsymbol{x},p)\end{aligned}$$

Solution of Continuity Equation

For continuity equation

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i(t, \boldsymbol{x}) + p \sum_{i=1}^{n} \frac{\partial F_i(t, \boldsymbol{x})}{\partial x_i} = 0$$

$$a(t, \boldsymbol{x}, p) = 1, \quad b_i(t, \boldsymbol{x}, p) = F_i(t, \boldsymbol{x}), \quad c(t, \boldsymbol{x}, p) = -p \sum_{i=1}^{n} \frac{\partial F_i(t, \boldsymbol{x})}{\partial x_i}.$$

Characteristic equations

$$\frac{dt}{ds} = 1$$

$$\frac{dx_i}{ds} = F_i(t, \boldsymbol{x})$$

$$\frac{dp}{ds} = -p \sum_{i=1}^{n} \frac{\partial F_i(t, \boldsymbol{x})}{\partial x_i}$$

$\dot{\boldsymbol{x}} = \boldsymbol{F}(t, \boldsymbol{x})$ evolution of $\boldsymbol{x}(t)$

$\dot{p} = -p(\boldsymbol{\nabla} \cdot \boldsymbol{F})$ evolution of p along $\boldsymbol{x}(t)$

Initial Conditions

- $\boldsymbol{x}_0$ Samples from $p(t_0, \boldsymbol{x})$
- $p_0 = p(t_0, \boldsymbol{x}_0)$ Values of $p(t_0, \boldsymbol{x})$ at $\boldsymbol{x}_0$

Parametric Uncertainty & Process Noise

Given system dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(t, \boldsymbol{x}, \boldsymbol{\Delta}) + \boldsymbol{n}(t, \omega)$$

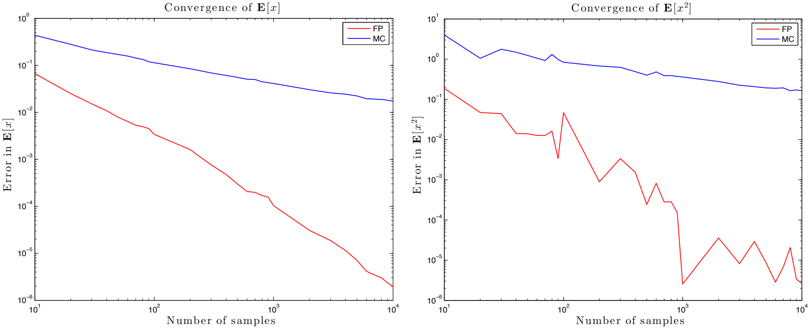
- Expand $n(t, \omega)$ using KL expansion.
- New paramters: $\boldsymbol{\xi} := (\boldsymbol{\xi}_0, \boldsymbol{\xi}_0^*, \cdots, \boldsymbol{\xi}_N, \boldsymbol{\xi}_N^*)^T$
- PDF: $p_{\boldsymbol{\xi}}(\boldsymbol{\xi})$
- Parameter PDF: $p_{\boldsymbol{\Delta}}(\boldsymbol{\Delta})$
- State IC PDF: $p_{\boldsymbol{x}}(t_0, \boldsymbol{x})$

Augment state space

$$\boldsymbol{X} := \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\Delta} \\ \boldsymbol{\xi} \end{pmatrix}, \qquad \text{with} \qquad \dot{\boldsymbol{X}} := \begin{pmatrix} \boldsymbol{G}(t, \boldsymbol{x}, \boldsymbol{\Delta}, \boldsymbol{\xi}) \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{H}(t, \boldsymbol{X})$$

with $p_{\boldsymbol{X}}(t_0, \boldsymbol{X}) := p_{\boldsymbol{x}}(t_0, \boldsymbol{x}) p_{\boldsymbol{\Delta}}(\boldsymbol{\Delta}) p_{\boldsymbol{\xi}}(\boldsymbol{\xi})$ and $p_{\boldsymbol{X}}(t, \boldsymbol{X}) := \mathcal{P}_t \, p_{\boldsymbol{X}}(t_0, \boldsymbol{X})$.

Better Accuracy & Faster Convergence than MC



(a) First moment   (b) Second moment

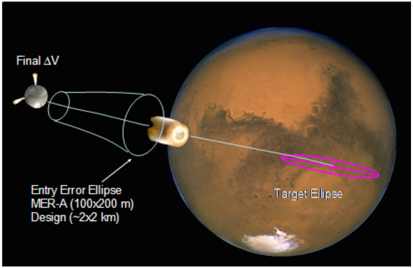
- Data generated from univariate normal distribution
- MC: PDF from kernel density estimation
- FP: PDF from spline interpolation
- Samples generated 1000 times for a given size. Plots show average error vs sample size

Requires $\frac{\partial F_i(\boldsymbol{x})}{\partial x_i}$.

Nonlinear Example

3 DOF Vinh's Equation
Models motion of spacecraft during planetary entry

$$
\begin{aligned}
\dot{h} &= V\sin(\gamma)\\
\dot{V} &= -\frac{\rho R_0}{2B_c}V^2 - \frac{gR_0}{v_c^2}\sin(\gamma)\\
\dot{\gamma} &= \frac{\rho R_0}{2B_c}\frac{C_L}{C_D}V + \frac{gR_0}{v_c^2}\cos(\gamma)\left(\frac{V}{R_0+h} - \frac{1}{V}\right).
\end{aligned}
$$



R_0 – radius of Mars

ρ – atmospheric density

v_c – escape velocity

$\frac{C_L}{C_D}$ – lift over drag

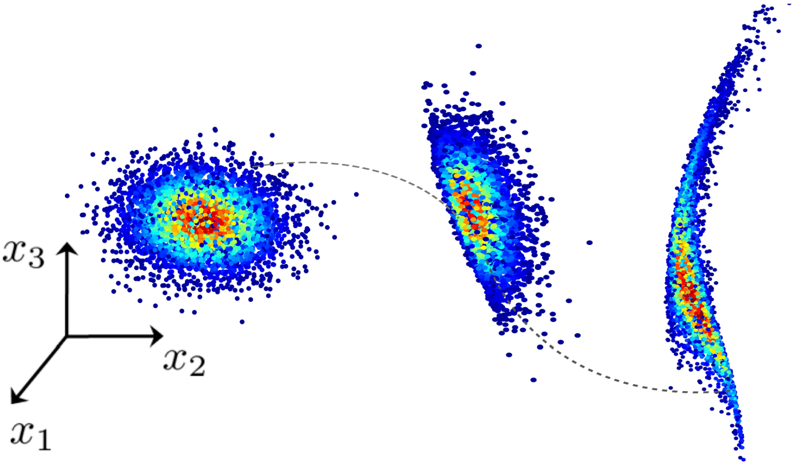
B_c – ballistic coefficient

h – height

V – velocity

γ – flight path angle

3DOF Vinh's Equation



- Gaussian initial condition uncertainty in (h, V, γ)

Frobenius-Perron Operator

Papers

1. A. Halder, R. Bhattacharya, *Beyond Monte Carlo: A Computational Framework for Uncertainty Propagation in Planetary Entry, Descent and Landing* , AIAA GNC 2010.
2. A. Halder, R. Bhattacharya, *Dispersion Analysis in Hypersonic Flight During Planetary Entry Using Stochastic Liouville Equation*, AIAA Journal of Guidance, Control, and Dynamics, 2011, 0731-5090 vol.34 no.2 (459-474)
3. P. Dutta, A. Halder, R. Bhattacharya, *Uncertainty Quantification for Stochastic Nonlinear Systems using Perron-Frobenius Operator and Karhunen-Loeve Expansion*. IEEE Multi-Conference on Systems and Control, Dubrovnik, Oct 2012.