

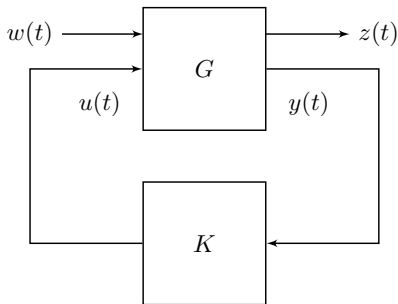
# **Output Feedback Control**

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# $\mathcal{H}_2$ Optimal Controller

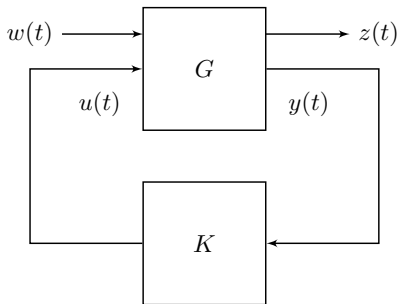
# $\mathcal{H}_2$ Optimal Controller



Let

$$\hat{G}(s) = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & \mathbf{0} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right], \quad \hat{K}(s) = \left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right].$$

# $\mathcal{H}_2$ Optimal Controller



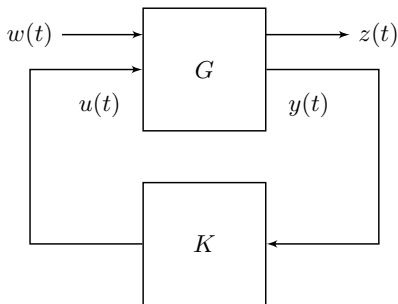
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$K(s)$  is essentially full-state feedback with  $\mathcal{H}_2$  estimator.

# $\mathcal{H}_\infty$ Optimal Controller

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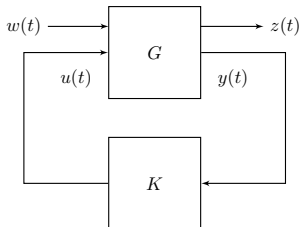
Let

$$\hat{G}(s) = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right], \quad \hat{K}(s) = \left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right].$$

Find  $K(s)$  that minimizes  $\|\hat{G}_{w \rightarrow z}\|_\infty$ .

# $\mathcal{H}_\infty$ Optimal Control

Simpler case with  $D_{22} = 0$



System

$$\hat{G}(s) = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right],$$

$$\hat{K}(s) = \left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right].$$

Closed-loop transfer function is

$$\begin{aligned} \mathcal{F}_l(\hat{G}, \hat{K}) &= \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}, \\ &= \left[ \begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array} \right]. \end{aligned}$$

# $\mathcal{H}_\infty$ Optimal Control

Simpler case with  $D_{22} = 0$  (contd.)

Define matrices

$$J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix},$$

and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$\bar{C} = [C_1 \quad 0], \quad \underline{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \quad \underline{D}_{12} = [0 \quad D_{12}]$$

$$\underline{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

These will be useful in reconstructing controller from LMI solution.



# $\mathcal{H}_\infty$ Optimal Control

Simpler case with  $D_{22} = 0$  (contd.)

Therefore

$$\begin{aligned} A_{cl} &= \bar{A} + \underline{B} \underline{J} \underline{C}, & B_{cl} &= \bar{B} + \underline{B} \underline{J} \underline{D}_{21} \\ C_{cl} &= \bar{C} + \underline{D}_{12} \underline{J} \underline{C}, & D_{cl} &= D_{11} + \underline{D}_{12} \underline{J} \underline{D}_{21}. \end{aligned}$$

# $\mathcal{H}_\infty$ Optimal Control

Simpler case with  $D_{22} = 0$  (contd.)

**Theorem**  $A_{cl}$  is Hurwitz and  $\|\mathcal{F}_l(\hat{G}, \hat{K})\|_\infty < \gamma$ , **iff** there exists symmetric matrices  $X > 0$  and  $Y > 0$  such that

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*X + XA & XB_1 & C_1^* \\ B_1^*X & -\gamma I & D_{11}^* \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*Y + YA & YC_1 & B_1^* \\ C_1^*Y & -\gamma I & D_{11}^* \\ B_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0,$$

where  $N_o$  and  $N_c$  are full rank matrices satisfying

$$\mathbf{Im} N_o = \mathbf{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \text{ and } \mathbf{Im} N_c = \mathbf{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}.$$

# $\mathcal{H}_\infty$ Optimal Control

Simpler case with  $D_{22} = 0$  (contd.)

**Proof:** See *A Linear Matrix Inequality Approach to  $\mathcal{H}_\infty$  Control* – Pascal Gahinet, Pierre Apkarian, 1994.

Main Ingredients:

- KYP Lemma
- Projection Lemmas

# $\mathcal{H}_\infty$ Controller Reconstruction

Suppose we have solved the LMIs and have obtained  $X, Y$ . Define  $X_{cl}$  as

$$X_{cl} = \begin{bmatrix} X & X_2^* \\ X_2 & I \end{bmatrix},$$

such that

$$X - Y^{-1} = X_2 X_2^*.$$

The LMI in the synthesis enforces

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$$

implies

$$X - Y^{-1} \geq 0.$$

Therefore, we can set

$$X_2 := \sqrt{X - Y^{-1}}. \text{ symmetric}$$

# $\mathcal{H}_\infty$ Controller Reconstruction

contd.

With  $X_2$  determined,  $X_{cl}$  is defined.

Define matrices,

$$P_{X_{cl}} = [\underline{B}^* \quad 0 \quad \underline{D}_{12}^*], \quad Q = [\underline{C} \quad \underline{D}_{21} \quad 0],$$
$$H_{X_{cl}} = \begin{bmatrix} \bar{A}^* X_{cl} + X_{cl} \bar{A} & X_{cl} \bar{B} & \bar{C}^* \\ \bar{B}^* X_{cl} & -\gamma I & D_{11}^* \\ \bar{C} & D_{11} & -\gamma I \end{bmatrix}.$$

# $\mathcal{H}_\infty$ Controller Reconstruction

contd.

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Controller matrix  $J$  can be obtained using reciprocal projection lemma (see Gahinet, Apkarian 1994)

$$H_{X_{cl}} + Q^* J^* P_{X_{cl}} + P_{X_{cl}} J Q < 0.$$

# $\mathcal{H}_\infty$ Controller Reconstruction

contd.

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- Many solutions for  $J$  exists!
- A family of  $\gamma$  – optimal  $\mathcal{H}_\infty$  controllers exists.
- Formulation is LMI feasibility

# Few Comments about $\mathcal{H}_\infty$

- We do not seek optimal  $\gamma$  as the computations become singular – instead we seek suboptimal  $\gamma$
- Bisection algorithm is used to reduced  $\gamma$  and LMIs are solved
- Issue with controller-plant pole-zero cancellation
- Poles in  $j\omega$  axis

LMI framework provides a flexibility to address/circumvent these issues

(see Gahinet, Apkarian 1994)



# Application of $\mathcal{H}_\infty$ Controller

$(V, \gamma)$  Tracking Controller for Longitudinal F16 Model

```
clc; clear;

% Define F16 Model
% =====
load f16LongiLinear.mat
wd = 1; % Scaling to adjust alpha disturbance

A = f16ss.a;
Bu = f16ss.b;
Bd = [0;wd;0;0];
B = [Bu Bd];
Cy = [1 0 0 0; % Velocity
      0 -1 1 0]; % gamma

[ns,nu] = size(B);
F16 = ss(A,B,Cy,zeros(2,nu));

% Define Weights
% =====
s = tf('s');

Wr = blkdiag(1/(s/1+1), 1/(s/5+1));
Wu = blkdiag(1/5000,1/25);
We = blkdiag(1, 1);
Wd = 0.1/(s/.1+1);
Wn = 0.01*blkdiag(1,1);
Wm = blkdiag(1, 1);

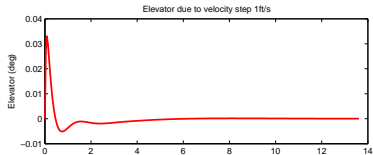
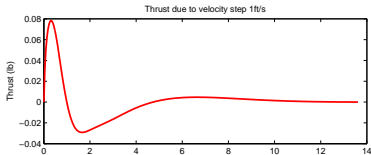
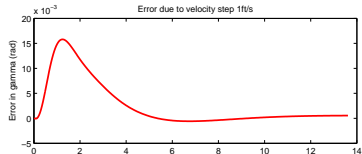
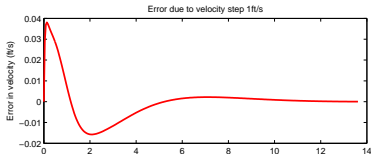
% System Interconnection
% =====
r = icsignal(2);
d = icsignal(1);
n = icsignal(2);
u = icsignal(2);
y = F16*[u;Wd*d];
e = (Wr*r - y);

G = iconnect;
G.input = [r;d;n;u];
G.output = [We*e;Wu*u;Wr*r-y-Wn*n];

[K,F16cl,gam,info] = hinfsyn(G.System,2,2,...
                             'method','lmi');
disp(sprintf('Minimum gamma = %f',gam));
```

# Application of $\mathcal{H}_\infty$ Controller

( $V, \gamma$ ) Tracking Controller for Longitudinal F16 Model



$$\gamma^* = 0.971881$$