

# Quadratic Stability of Dynamical Systems

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# Quadratic Lyapunov Functions

# Quadratic Stability

Dynamical system

$$\dot{x} = Ax,$$

is quadratically stable if

$$\exists V(x) \geq 0, \quad \dot{V} \leq 0.$$

Let  $V(x) = x^T P x$ ,  $P \in \mathbb{S}_{++}^n$  ( $P = P^T > 0$ )

Therefore,

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + P A) x\end{aligned}$$

Therefore

$$\dot{V} \leq 0 \implies x^T (A^T P + P A) x \leq 0 \implies A^T P + P A \leq 0.$$

# Lyapunov Equation

We can write

$$A^T P + P A \leq 0$$

as

$$A^T P + P A + Q = 0$$

for  $Q = Q^T \geq 0$ .

## Interpretation

For linear system  $\dot{x} = Ax$ , if  $V(x) = x^T P x$ ,

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P (Ax) \\ &= -x^T Q x.\end{aligned}$$

If  $V(x) = x^T P x$  is **generalized energy**,  $\dot{V} = -x^T Q x$  is **generalized dissipation**.

# Stability Condition

If  $P > 0, Q > 0$ , then  $\dot{x} = Ax$

- is **globally asymptotically stable**
- $\Re \lambda_i(A) < 0$

Note that for  $P = P^T > 0$ , eigenvalues are real

$$\begin{aligned} \implies \lambda_{\min}(P) x^T x &\leq x^T P x \leq \lambda_{\max}(P) x^T x \\ \implies \dot{V} = -x^T Q x &\leq -\lambda_{\min}(Q) x^T x \\ &\leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} x^T P x \\ &= -\alpha V(x) \end{aligned}$$

# Lyapunov Integral

If  $A$  is stable, then

$$P = \int_0^{\infty} e^{tA^T} Q e^{tA} dt, \text{ for any } Q = Q^T > 0.$$

**Proof:**

Substitute it in LHS of Lyapunov equation to get,

$$\begin{aligned} A^T P + P A &= \int_0^{\infty} \left( A^T e^{tA^T} Q e^{tA} + e^{tA^T} Q e^{tA} A \right) dt, \\ &= \int_0^{\infty} \left( \frac{d}{dt} e^{tA^T} Q e^{tA} \right) dt, \\ &= e^{tA^T} Q e^{tA} \Big|_0^{\infty}, \\ &= -Q. \end{aligned}$$

# Computation of $\|x\|_{2,Q}$

Recall

$$\|x\|_2^2 := \int_0^\infty x^T x dt.$$

Define weighted norm as

$$\|x\|_{2,Q}^2 := \int_0^\infty x^T Q x dt.$$

If  $x(t)$  is solution of  $\dot{x} = Ax$ ,

$$x(t) := e^{tA} x_0.$$

Substituting we get,

$$\begin{aligned} \|x\|_{2,Q}^2 &= \int_0^\infty x_0^T e^{tA^T} Q e^{tA} dt \\ &= x_0^T P x_0 \text{ assuming } A \text{ is stable} \end{aligned}$$

Cost-to-go interpretation

# LQR Problem



# Linear Quadratic Regulator

## Problem Statement

Given system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Determine  $u^*(t)$  that solves

$$\min_{u(t)} \|y\|_2 \text{ with } x(0) = x_0.$$

Or

$$\begin{aligned} \min_{u(t)} J &:= \int_0^{\infty} y^T y \, dt \\ &= \int_0^{\infty} (x^T C^T C x + x^T C^T D u + u^T D^T C x + u^T D^T D u) \, dt \\ &= \int_0^{\infty} (x^T C^T C x + u^T D^T D u) \, dt. \end{aligned}$$

# Linear Quadratic Regulator

*Solution as Optimal Control Problem*

$$\min_u \int_0^{\infty} (x^T Q x + u^T R u) dt, \quad Q = Q^T \geq 0, R = R^T > 0$$

subject to

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ x(0) &= x_0.\end{aligned}$$

- Euler Lagrange Equations
- Hamilton-Jacobi-Bellman Equation – Dynamic Programming

# **Euler Langrange Formulation**

# Linear Quadratic Regulator

*Solution as Optimal Control Problem – EL Formulation*

$$\min_u \int_0^T L(x, u) dt + \Phi(x(T)), \text{ subject to } \dot{x} = f(x, u).$$

Define  $H = L + \lambda^T f$ .

## Euler-Lagrange Equations

$$H_u = 0 \quad \dot{\lambda}^T = -H_x \quad \lambda(T) = \phi_x(x(T))$$

## Our Problem

$$\min_u \int_0^T (x^T Q x + u^T R u) dt, \text{ subject to } \dot{x} = A x + B u.$$

Define  $H = x^T Q x + u^T R u + \lambda^T (A x + B u)$ .

# Linear Quadratic Regulator

*Solution as Optimal Control Problem – EL Formulation*

## Our Problem

$$\min_u \frac{1}{2} \int_0^T (x^T Q x + u^T R u) dt, \text{ subject to } \dot{x} = Ax + Bu.$$

Define  $H = \frac{1}{2} (x^T Q x + u^T R u) + \lambda^T (Ax + Bu)$ .

## EL Equations

$$(1) \quad H_u = 0 \implies u^T R + \lambda^T B = 0 \implies u = -R^{-1} B^T \lambda.$$

$$(2) \quad \dot{\lambda}^T = -H_x = -x^T Q - \lambda^T A$$

$$(3) \quad \lambda(T) = 0.$$

# Linear Quadratic Regulator

*Solution as Optimal Control Problem – EL Formulation*

Let  $\lambda(t) = P(t)x(t)$

$$\begin{aligned} \implies \dot{\lambda} &= \dot{P}x + P\dot{x} \\ &= \dot{P}x + P(Ax + Bu), \\ &= \dot{P}x + P(Ax - BR^{-1}B^T Px), \\ &= (\dot{P} + PA - PBR^{-1}B^T P)x. \end{aligned}$$

From EL(2) we get

$$\begin{aligned} \dot{\lambda} &= -Qx - A^T Px \\ \implies (\dot{P} + PA + A^T P - PBR^{-1}B^T P + Q)x &= 0 \\ \implies \dot{P} + PA + A^T P - PBR^{-1}B^T P + Q &= 0. \text{ Riccati Differential Equation} \end{aligned}$$

In the steady-state  $T \rightarrow \infty$ ,  $\dot{P} = 0$ ,

$$PA + A^T P - PBR^{-1}B^T P + Q = 0. \text{ Algebraic Riccati Equation}$$

# Linear Quadratic Regulator

*Solution as Optimal Control Problem – EL Formulation*

$$\min_u \frac{1}{2} \int_0^{\infty} (x^T Q x + u^T R u) dt, \text{ subject to } \dot{x} = Ax + Bu.$$

is equivalent to

$$PA + A^T P - PBR^{-1}B^T P + Q = 0,$$
$$u = -R^{-1}B^T P.$$

# Hamilton-Jacobi-Bellman Formulation



# Hamilton-Jacobi-Bellman Approach

Let

$$V^*(x(t)) = \min_{u[t,\infty)} \frac{1}{2} \int_t^\infty (x^T Q x + u^T R u) dt$$

subject to

$$\dot{x} = Ax + Bu.$$

# Hamilton-Jacobi-Bellman Approach

*contd.*

$$\begin{aligned} V^*(x(t)) &= \min_{u[t,\infty)} \frac{1}{2} \int_t^\infty (x^T Q x + u^T R u) dt \\ &= \min_{u[t,t+\Delta t]} \left\{ \int_t^{t+\Delta t} \frac{1}{2} (x^T Q x + u^T R u) dt + V^*(x(t + \Delta t)) \right\} \end{aligned}$$

Let  $V(x) := x^T P x$ , therefore,

$$\begin{aligned} V^*(x(t)) &= \min_{u[t,t+\Delta t]} \left\{ \frac{1}{2} (x^T Q x + u^T R u) \Delta t + V^*(x(t)) + \right. \\ &\quad \left. (Ax + Bu)^T P x \Delta t + x^T P (Ax + Bu) \Delta t + H.O.T \right\} \end{aligned}$$

# Hamilton-Jacobi-Bellman Approach

*contd.*

$$\implies \min_{u[t, t+\Delta t]} \left\{ \frac{1}{2} (x^T Q x + u^T R u) + (Ax + Bu)^T P x + x^T P (Ax + Bu) + H.O.T \right\} = 0.$$

$$\lim_{\Delta t \rightarrow 0} \implies \min_u \left\{ \frac{1}{2} (x^T Q x + u^T R u) + (Ax + Bu)^T P x + x^T P (Ax + Bu) \right\} = 0$$

Quadratic in  $u$ ,

$$\implies u^* = -R^{-1} B^T P x.$$

Optimal controller is state-feedback.

# **Variational Approach**

# A Third Approach

Given dynamics

$$\dot{x} = Ax + Bu,$$

with controller  $u = Kx$ , find  $K$  that minimizes

$$J := \frac{1}{2} \int_0^{\infty} (x^T Q x + u^T R u) dt = \frac{1}{2} \int_0^{\infty} x^T (Q + K^T R K) x dt.$$

The closed-loop dynamics is

$$\dot{x} = Ax + Bu = (A + BK)x = A_c x.$$

The solution is therefore,

$$x(t) = e^{tA_c} x_0.$$

The cost function is therefore,

$$J := \frac{1}{2} x_0^T \left( \int_0^{\infty} e^{tA_c^T} (Q + K^T R K) e^{tA_c} dt \right) x_0 = \frac{1}{2} x_0^T P x_0.$$

# A Third Approach

*contd.*

Apply the following 'trick'

$$\int_0^{\infty} \frac{d}{dt} e^{tA_c^T} (Q + K^T R K) e^{tA_c} dt =$$

$$A_c^T \left( \int_0^{\infty} e^{tA_c^T} (Q + K^T R K) e^{tA_c} dt \right)$$

$$+ \left( \int_0^{\infty} e^{tA_c^T} (Q + K^T R K) e^{tA_c} dt \right) A_c$$

$$\left[ e^{tA_c^T} (Q + K^T R K) e^{tA_c} \right]_0^{\infty} = A_c^T P + P A_c$$

Or

$$A_c^T P + P A_c + Q + K^T R K = 0,$$

Or

$$(A + BK)^T P + P(A + BK) + Q + K^T R K = 0.$$

# A Third Approach

*contd.*

The optimal cost is therefore,

$$J^* = \frac{1}{2}x_0^T P^* x_0 \implies \left. \frac{\partial J}{\partial P} \right|_{P^*} = 0.$$

- Variation  $\delta P$  from  $P^*$  should result in  $\delta J = 0$
- Let  $P = P^* + \delta P$ ,  $\implies J = \frac{1}{2}x_0^T P^* x_0 + \underbrace{\frac{1}{2}x_0^T \delta P x_0}_{\delta J}$
- $\delta J = 0 \implies \delta P = 0$

# A Third Approach

*contd.*

Substitute  $P = P^* + \delta P$ , and  $K = K^* + \delta K$  in the equality constraint

$$(A + BK)^T P + P(A + BK) + Q + K^T R K = 0,$$

to get,

$$(A + B(K^* + \delta K))^T (P^* + \delta P) + (P^* + \delta P)(A + B(K^* + \delta K)) \\ + Q + (K^* + \delta K)^T R (K^* + \delta K) = 0,$$

or

$$(A + BK^*)^T P^* + P^*(A + BK^*) + Q + K^{*T} R K^* + \\ \delta P(A + BK^*) + (*)^T + \delta K^T (B^T P^* + R K^*) + (*)^T \\ + H.O.T = 0.$$

$$\implies K^* = -R^{-1} B^T P^*.$$



# **LMI Formulation**

# Problem Formulation

Find gain  $K$  such that  $u = Kx$  minimizes

$$\int_0^{\infty} (x^T Q x + u^T R u) dt,$$

subject to dynamics

$$\dot{x} = Ax + Bu,$$

and

$$x(0) = x_0.$$

# An Upper Bound on the Cost-to-go

If  $\exists V(x) > 0$  such that

$$\frac{dV}{dt} \leq -(x^T Q x + u^T R u).$$

Integrating from  $[0, T]$ , gives us

$$\int_0^T \frac{dV}{dt} dt \leq - \int_0^T (x^T Q x + u^T R u) dt,$$

or

$$V(x(T)) - V(x(0)) \leq - \int_0^T (x^T Q x + u^T R u) dt.$$

Since  $V(x(T)) \geq 0$  for any  $T$

$$\implies -V(x(0)) \leq - \int_0^T (x^T Q x + u^T R u) dt,$$

# An Upper Bound on the Cost-to-go

Since  $V(x(T)) \geq 0$  for any  $T$

$$\implies -V(x(0)) \leq -\int_0^T (x^T Qx + u^T Ru) dt,$$

or

$$V(x(0)) \geq \int_0^\infty (x^T Qx + u^T Ru) dt.$$

## Sufficient condition for upper-bound on cost-to-go.

If  $\exists V(x) > 0$  such that

$$\frac{dV}{dt} \leq -(x^T Qx + u^T Ru).$$

**Idea:**

Minimize upper-bound to get optimal  $K$ .

# Optimization Problem

Find  $P = P^T > 0$  and  $K$  such that with  $V := x^T P x$ ,

$$\min_{P,K} V(x(0)) = x(0)^T P x(0) \quad \text{Cost Function}$$

subject to

$$\dot{V} \leq -x^T (Q + K^T R K) x \quad \text{Constraint Function}$$

**Or equivalently**

$$\min_{P,K} \text{tr} P$$

subject to

$$(A + BK)^T P + P(A + BK) + Q + K^T R K \leq 0.$$

# Optimization Problem

$$\min_{P,K} \text{tr} P$$

subject to

$$(A + BK)^T P + P(A + BK) + Q + K^T R K \leq 0.$$

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