AERO 422: Active Controls for Aerospace Vehicles

Root Locus Design Method

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Root Locus
Root Locus

Generalized Setting

- Write

\[ 1 + P(s)C(s) = 1 + KL(s) = 0 \]

- Roots depend on \( K \) generalized gain

\[ 1 + KL(s) = 0 \]
\[ 1 + K \frac{N(s)}{D(s)} = 0 \]
\[ D(s) + KN(s) = 0 \]

or \( L(s) = -\frac{1}{K} \) root-locus form
Simple Example

\[ P(s) = \frac{A}{s(s + c)}, \quad C(s) = 1 \]

- Two roots
- Depends on parameters \( A \) and \( c \)

\[ r_1, r_2 = \frac{1}{2} \pm \frac{\sqrt{1 - 4A}}{2} \quad r_1, r_2 = \frac{c}{2} \pm \frac{\sqrt{c^2 - 4}}{2} \]

- RL studies variation of \( r_1, r_2 \) with respect to \( A, c \) one at a time
- MATLAB command \texttt{rlocus(...)} is used to generate these plots
- help \texttt{rlocus} for more details
Simple Example

Variation w.r.t $A$

- Study variation w.r.t $A$, set $c = 1$

$$P(s) = \frac{A}{s(s + c)}, C(s) = 1$$

- Root-locus form

$$1 + PC = 0 \implies 1 + A \frac{1}{s(s + 1)} = 0$$

$$\text{or } \frac{1}{s(s + 1)} = -\frac{1}{A}$$
Simple Example

Variation w.r.t $A$ (contd.)

$$1 + A \frac{1}{s(s + 1)} = 0$$

MATLAB Code

```matlab
s = tf('s');
sys = 1/s/(s+1);
rlocus(sys);
```

$$r_1, r_2 = \frac{1}{2} \pm \frac{\sqrt{1 - 4A}}{2}$$
Simple Example

Variation w.r.t $c$

- Study variation w.r.t $c$, set $A = A^* = 1$

\[
P(s) = \frac{1}{s(s + c)}, C(s) = 1\]

- Root-locus form

\[
1 + PC = 0 \implies 1 + \frac{1}{s(s + c)} = 0
\]

or $s^2 + cs + 1 = 0$

or $(s^2 + 1) + cs = 0$

or $L'(s) = \frac{s}{s^2 + 1} = -\frac{1}{c}$
Simple Example

Variation w.r.t \( c \) (contd.)

\[ \frac{s}{s^2 + 1} = -\frac{1}{c} \]

MATLAB Code

```matlab
s = tf('s');
sys = s/(s^2+1);
rlocus(sys);
```

\[ r_1, r_2 = \frac{c}{2} \pm \frac{\sqrt{c^2 - 4}}{2} \]
Guidelines for Plotting Root Locus
Guidelines for Drawing Root Locus

Definition 1
Root locus of $L(s)$ is the set of values of $s$ for which
$$1 + KL(s) = 0$$
for values of $0 \leq K < \infty$.

Definition 2
Root locus is the set of values of $s$ for which phase of $L(s)$ is $180^\circ$.
Let the angle from a zero be $\psi_i$ and angle from a pole be $\phi_i$. Then
$$\sum_j \psi_j - \sum_i \phi_i = 180^\circ + 360^\circ (l - 1)$$
for integer $l$. 
Step 1

Draw poles and zeros of $L(s)$

Given $P(s) = \frac{1}{s[(s+4)^2+16]}$, $C(s) = K$.

Plot poles with $\times$

0, $-4 \pm 4j$ for this example

Plot zeros with $O$

None for this example
Step 2

Real axis portions of the locus

If we take $s_0$ on the real-axis
- contributions from complex poles and zeros disappear
- Angle criterion :
  \[ \sum_j \psi_j - \sum_i \phi_i = 180^\circ + 360^\circ (l - 1) \]
  \[ \phi_1 = -\phi_2 \]
  \[ \sqrt{-4 + 4j} = -\sqrt{-4 - 4j} \]
- $s_0$ must lie to the left of odd number of real poles & zeros
**Step 2**

*Real axis portions of the locus (contd.)*

Let there be
- a pole at $-2$ and
- a zero at $-4$

How does the root locus change?
Step 3

Asymptotes

Study behavior for large $K$,

$$L(s) = -\frac{1}{K}$$

$$K \to \infty \implies L(s) = 0$$

For large values of $K$, roots will be close to zeros of $L(s)$.

- But there are $n$ poles and $m$ zeros, with $n > m$.
- Where do $n - m$ poles go?

They are asymptotic to lines with angles $\phi_r$ starting from $s = \alpha$, where

$$\phi_r = \frac{180^\circ + 360^\circ (r - 1)}{n - m}, \quad \alpha = \frac{\sum p_i - \sum z_j}{n - m}.$$
Step 3

Asymptotes (contd.)

For this example

\[ n = 3, m = 0 \implies \alpha = 60^\circ, 180^\circ, 300^\circ, \]

and \( \alpha = -2.67 \).
Step 4

Departure Angles

Angle at which a branch of locus departs from one of the poles

\[ r\phi_{\text{dep}} = \sum \psi_i - \sum \phi_j - 180^\circ - 360^\circ r, \]

where \( \sum \phi_j \) is over the other poles.

We assume there multiple poles of order \( q \) under consideration, and \( r = 1, \cdots, q. \)

Summation \( \sum \psi_j \) is over all zeros.
Step 4

Arrival Angles

Angle at which a branch of locus arrives at one of the zeros

\[ r\psi_{\text{arr}} = \sum \phi_j - \sum \psi_i + 180^\circ + 360^\circ r, \]

where \( \sum \psi_j \) is over the other zeros.

We assume there \textbf{multiple zeros of order} \( q \) under consideration, and \( r = 1, \cdots, q \).

Summation \( \sum \phi_j \) is over all poles.
Step 4

Example
Step 5

*Imaginary axis crossing*

- Use Routh’s table to determine $K$ for stability for

$$s^3 + 8s^2 + 32s + K = 0,$$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td>8</td>
<td>$K$</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$32 - K/8$</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>$K$</td>
<td>0</td>
</tr>
</tbody>
</table>

- $K > 0$ and $32 - K/8 > 0 \implies K > 256$

- Root locus crosses imaginary axis for $K = 256$.

- Substitute $K = 256$ and $s = j\omega_0$ in characteristic equation, and solve for $\omega_0$.

$$(j\omega_0)^3 + 8(j\omega_0)^2 + 32(j\omega_0) + 256 = 0$$
Step 5

Imaginary axis crossing (contd.)

- Solve for $\omega_0$

\[(j\omega_0)^3 + 8(j\omega_0)^2 + 32(j\omega_0) + 256 = 0\]

\[\implies -8\omega_0^2 + 256 = 0, \text{ and } -\omega_0^3 + 32\omega_0 = 0.\]

or $\omega_0 = \pm \sqrt{32}.$
Step 6

Arrival & departure angles at multiple root locations

Few examples

- Two segments come together at 180° and break away at ±90°
- Three locus segments approach at relative angles of 120° and depart at angles rotated by 60°
- Read textbook for details
Final Plot

Root Locus

Plotting Guidelines

Dynamic Compensator

Design Example
Dynamic Compensators
Lead Compensator

Stabilizing effect

Compensator form

\[ C(s) = K \frac{s/z + 1}{s/p + 1} \]

- \( p >> z > 0 \): \( p \) not too far to the left
- Root locus:
  \[ \frac{s/z + 1}{s/p + 1} P(s) = -\frac{1}{K} \]
  - Moves the locus to the left
Lead Compensator

Example

Plant model $P(s) = \frac{1}{s(s+1)}$

(a) $C(s) = \frac{s}{2} + 1$

(b) $C(s) = \frac{s/2+1}{s/15+1}$

(c) $C(s) = \frac{s/2+1}{s/10+1}$

- Root Locus: $C(s)P(s) = -\frac{1}{K}$
- Location of $z$, $p$ is based on trial and error
- Select desired closed-loop pole
  - Arbitrarily pick $z$, then use angle criterion to select $p$
Lag Compensator

*Improves steady state performance*

Compensator form

\[ C(s) = K \frac{s + z}{s + p} \]

- \( z > p > 0 \) low frequency, near the origin
- \( z \) is close to \( p \)
- Root locus:

\[ \frac{s + z}{s + p} P(s) = -\frac{1}{K} \]

- Boosts steady-state gain: \( z/p > 1 \)
Lag Compensator

Example

- Plant: \( \frac{1}{s(s+1)} \), Lead Compensator: \( \frac{K(s+2)}{s+15} \)

- \( K_v := \lim_{s \to 0} s \frac{K(s+2)}{s+15} \frac{1}{s(s+1)} = 90 \times \frac{2}{15} = 12 \).

- Steady-state to ramp input = \( \frac{1}{K_v} = \frac{1}{12} = 0.0833 \)

- How to increase \( K_v \)? reduce \( e_{ss} \) to ramp

- Introduce a lag compensator: \( \frac{s+0.05}{s+0.01} \)

- \( K_v := \lim_{s \to 0} s \frac{K(s+0.05)}{s+0.01} \frac{s+2}{s+15} \frac{1}{s(s+1)} = 5 \times 12 = 60 \)

- Steady-state to ramp input = \( \frac{1}{K_v} = \frac{1}{60} = 0.0166 \)

**Lag compensators amplify gain at low frequency**
**Have no effect at high-frequency**
Lag Compensator

Caution

- Closed-loop poles are near the zero at $-0.05$
- Very slow decay rate.
- Proximity of poles $\rightarrow$ low amplitude
- May affect settling time, especially for disturbance response

Put lag pole-zero at as high frequency possible, without affecting transients
Design Example
Control of a Small Airplane

*Piper Dakota (from text book)*

**System**
Transfer function from $\delta_e$ (elevator angle) to $\theta$ (pitch angle) is

$$P(s) = \frac{\theta(s)}{\delta_e(s)} = \frac{160(s + 2.5)(s + 0.7)}{(s^2 + 5s + 40)(s^2 + 0.03s + 0.06)}$$

**Control Objective 1**
Design an autopilot so that the step response to elevator input has $t_r < 1$ and $M_p < 10\% \implies \omega_n > 1.8$ rad/s and $\zeta > 0.6$ $2^{nd}$ order
Control of a Small Airplane

- Open Loop Poles: \(-2.5 \pm 5.81j, -0.015 \pm 0.244j\) (stable)
- Open Loop Zeros: \(-2.5, -0.7\) (no RHS zeros)

Figure: Root locus with proportional feedback
Control of a Small Airplane

Proportional Controller

Not possible to satisfy $\zeta$ requirement with just proportional controller
Control of a Small Airplane

Lead Compensator

After trial and error, choose \( C(s) = K \frac{s+3}{s+25} \), with \( K = 1.5 \)

(a) Step response with lead compensator

(b) Root locus with lead compensator

Has steady-state error ... have to fix this.
Control of a Small Airplane

*Lead Compensator + Integral Control*

**Fix Steady-State Error**

- introduce integral control

\[ C(s) = KD_c(s)(1 + K_I/s) \]

- tune \( K_I \) to get desired behaviour
- study root locus w.r.t \( K_I \)

**Characteristic Equation**

\[ 1 + KD_c(s)P(s) + \frac{K_I}{s}KD_c(s)P(s) = 0 \]

Write this in \( L(s) = -\frac{1}{K_I} \) form
Control of a Small Airplane

Lead Compensator + Integral Control (contd.)

Characteristic Equation

\[ 1 + KD_c(s)P(s) + \frac{KI}{s} KD_c(s)P(s) = 0 \]

Write this in \( L(s) = -\frac{1}{KI} \) form

\[ L(s) = \frac{1}{s} \frac{KD_cP}{1 + KD_cP} \]
Control of a Small Airplane

Lead Compensator + Integral Control (contd.)

(a) Root locus with PI

(b) Root locus with PI (zoomed)

For $K_I > 0$, $\zeta \downarrow \implies M_p \uparrow$
Control of a Small Airplane

*Lead Compensator + Integral Control (contd.)*

- Choose small value of $K_I = 0.15$
- Higher overshoot at the cost of zero steady-state error

(a) Root locus with PI

(b) Root locus with PI (zoomed)
Control of a Small Airplane – Analysis

Control $u(t)$

- High frequency in $u(t)$ is undesirable (rate limit & controller roll off)
- Large values for $u(t)$ is undesirable (saturation)
Control of a Small Airplane – Analysis

How good is this controller?

![Graphs showing control system analysis](image-url)