AERO 422: Active Controls for Aerospace Vehicles

Root Locus Design Method

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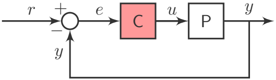
Root Locus

Root Locus

Generalized Setting



- Write

$$1 + P(s)C(s) = 1 + KL(s) = 0$$

- Roots depend on K generalized gain

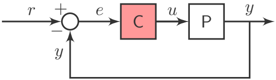
$$1 + KL(s) = 0$$

$$1 + K\frac{N(s)}{D(s)} = 0$$

$$D(s) + KN(s) = 0$$

$$\text{or } L(s) = -\frac{1}{K}$$ root-locus form

Simple Example



$$P(s) = \frac{A}{s(s+c)}, C(s) = 1$$

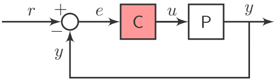
- Two roots
- Depends on parameters A and c

$$r_1, r_2 = \frac{1}{2} \pm \frac{\sqrt{1-4A}}{2} \qquad r_1, r_2 = \frac{c}{2} \pm \frac{\sqrt{c^2-4}}{2}$$

- RL studies variation of r_1, r_2 with respect to A, c one at a time
- `MATLAB` command `rlocus(...)` is used to generate these plots
- `help rlocus` for more details

Simple Example

Variation w.r.t A



- Study variation w.r.t A, set $c = 1$

$$P(s) = \frac{A}{s(s+c)}, C(s) = 1$$

- Root-locus form

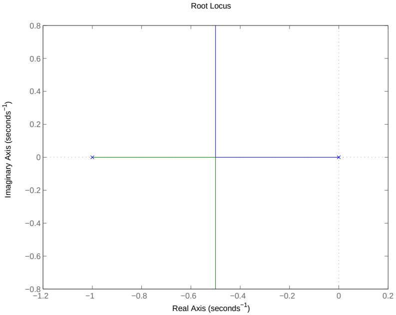
$$1 + PC = 0 \implies 1 + A\frac{1}{s(s+1)} = 0$$

$$\text{or } \frac{1}{s(s+1)} = -\frac{1}{A}$$

Simple Example

Variation w.r.t A (contd.)

$$1 + A\frac{1}{s(s+1)} = 0$$



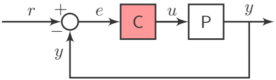
MATLAB Code

```
s = tf('s');
sys = 1/s/(s+1);
rlocus(sys);
```

$$r_1, r_2 = \frac{1}{2} \pm \frac{\sqrt{1-4A}}{2}$$

Simple Example

Variation w.r.t c



- Study variation w.r.t c, set $A = A^* = 1$

$$P(s) = \frac{1}{s(s+c)}, C(s) = 1$$

- Root-locus form

$$1 + PC = 0 \implies 1 + \frac{1}{s(s+c)} = 0$$

$$\text{or } s^2 + cs + 1 = 0$$

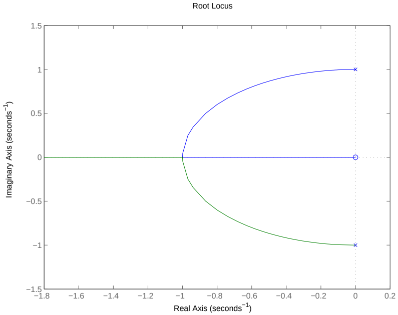
$$\text{or } (s^2 + 1) + cs = 0$$

$$\text{or } L'(s) = \frac{s}{s^2+1} = -\frac{1}{c}$$

Simple Example

Variation w.r.t c (contd.)

$$\frac{s}{s^2+1} = -\frac{1}{c}$$



MATLAB Code

```
s = tf('s');
sys = s/(s^2+1);
rlocus(sys);
```

$$r_1, r_2 = \frac{c}{2} \pm \frac{\sqrt{c^2-4}}{2}$$

Guidelines for Plotting Root Locus

Guidelines for Drawing Root Locus

Definition 1

Root locus of $L(s)$ is the set of values of s for which $1 + KL(s) = 0$ for values of $0 \leq K < \infty$.

Definition 2

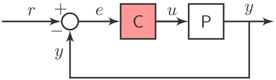
Root locus is the set of values of s for which phase of $L(s)$ is 180°. Let the angle from a zero be ψ_i and angle from a pole be ϕ_i. Then

$$\sum_j \psi_j - \sum_i \phi_i = 180^\circ + 360^\circ(l - 1)$$

for integer l.

Step 1

Draw poles and zeros of $L(s)$



Given $P(s) = \frac{1}{s[(s+4)^2+16]}$,

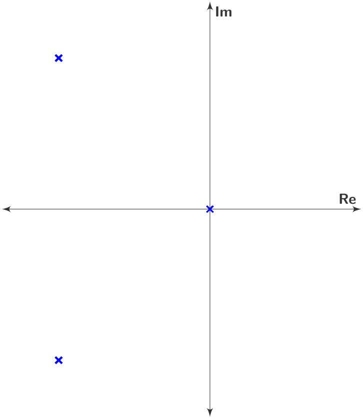
$C(s) = K$.

Plot poles with $\times$

$0, -4 \pm 4j$ for this example

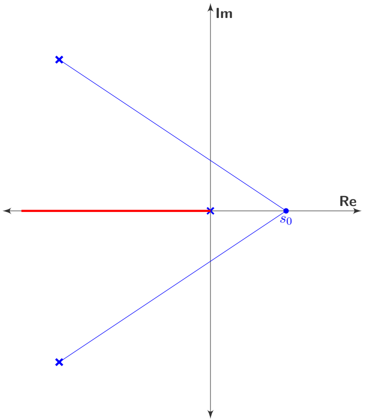
Plot zeros with O

None for this example



Step 2

Real axis portions of the locus



If we take s_0 on the real-axis

- contributions from complex poles and zeros disappear
- Angle criterion :

$$\sum_j \psi_j - \sum_i \phi_i = 180^\circ + 360^\circ(l-1)$$

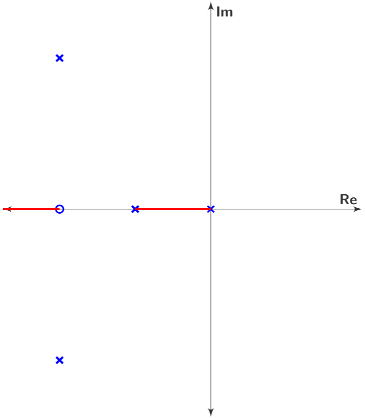
- $\phi_1 = -\phi_2$

$$\angle -4+4j = -\angle -4-4j$$

- s_0 must lie to the left of odd number of real poles & zeros

Step 2

Real axis portions of the locus (contd.)



Let there be

- a pole at -2 and
- a zero at -4

How does the root locus change?

Step 3

Asymptotes

Study behavior for large K,

$$L(s) = -\frac{1}{K}$$

$$K \to \infty \implies L(s) = 0$$

For large values of K, roots will be close to zeros of $L(s)$.

- But there are n poles and m zeros, with $n > m$.
- Where do $n - m$ poles go?

They are asymptotic to lines with angles ϕ_r starting from $s = \alpha$, where

$$\phi_r = \frac{180^\circ + 360^\circ(r-1)}{n-m}, \ \alpha = \frac{\sum p_i - \sum z_j}{n-m}.$$

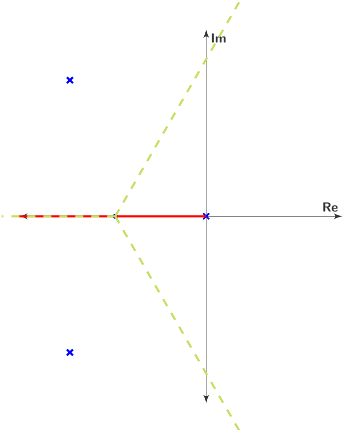
Step 3

Asymptotes (contd.)

For this example

$$n = 3, m = 0 \implies \alpha = 60^\circ, {\color{red}180^\circ}, 300^\circ,$$

and $\alpha = -2.67$.



Step 4

Departure Angles

Angle at which a branch of locus departs from one of the poles

$$r\phi_{\text{dep}} = \sum \psi_i - \sum \phi_j - 180^\circ - 360^\circ r,$$

where $\sum \phi_j$ is over the other poles.

We assume there multiple poles of order q under consideration, and $r = 1, \cdots, q$.

Summation $\sum \psi_j$ is over all zeros.

Step 4

Arrival Angles

Angle at which a branch of locus arrives at one of the zeros

$$r\psi_{\text{arr}} = \sum \phi_j - \sum \psi_i + 180^\circ + 360^\circ r,$$

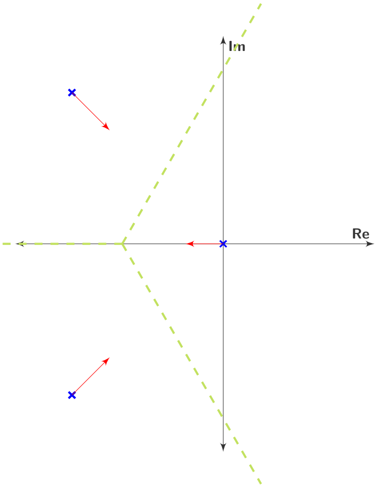
where $\sum \psi_j$ is over the other zeros.

We assume there multiple zeros of order q under consideration, and $r = 1, \cdots, q$.

Summation $\sum \phi_j$ is over all poles.

Step 4

Example



Step 5

Imaginary axis crossing

- Use Routh's table to determine K for stability for

$$s^3 + 8s^2 + 32s + K = 0,$$

$$\begin{array}{c|cc} s^3 & 1 & 32 \\ s^2 & 8 & K \\ s^1 & 32 - K/8 & 0 \\ s^0 & K & 0 \end{array}$$

- $K > 0$ and $32 - K/8 > 0 \implies K > 256$
- Root locus crosses imaginary axis for $K = 256$.
- Substitute $K = 256$ and $s = j\omega_0$ in characteristic equation, and solve for ω_0.

$$(j\omega_0)^3 + 8(j\omega_0)^2 + 32(j\omega_0) + 256 = 0$$

Step 5

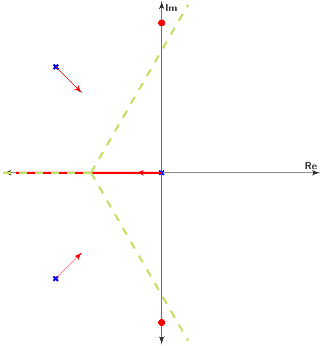
Imaginary axis crossing (contd.)

- Solve for ω_0

$$(j\omega_0)^3 + 8(j\omega_0)^2 + 32(j\omega_0) + 256 = 0$$

$$\implies -8\omega_0^2 + 256 = 0, \text{ and } -\omega_0^3 + 32\omega_0 = 0.$$

or $\omega_0 = \pm\sqrt{32}$.



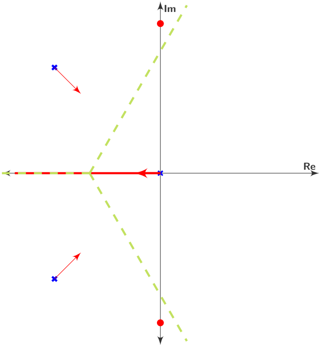
Step 6

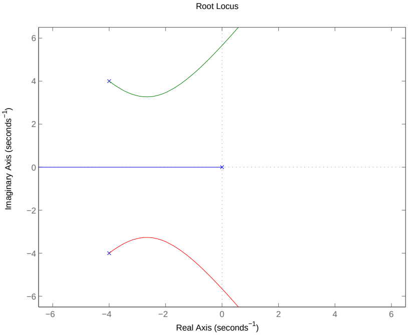
Arrival & departure angles at multiple root locations

Few examples

- Two segments come together at 180° and break away at $\pm 90^\circ$
- Three locus segments approach at relative angles of 120° and depart at angles rotated by 60°
- Read textbook for details

Final Plot

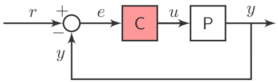

Im
Re

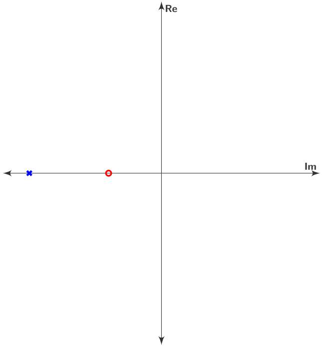

Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
6
4
2
0
−2
−4
−6

Dynamic Compensators

Lead Compensator

Stabilizing effect





Compensator form

$$C(s) = K\frac{s/z+1}{s/p+1}$$

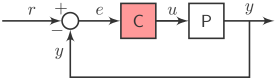
- $p >> z > 0$ p not too far to the left
- Root locus:

$$\frac{s/z+1}{s/p+1}P(s) = -\frac{1}{K}$$

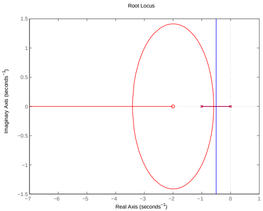
- Moves the locus to the left

Lead Compensator

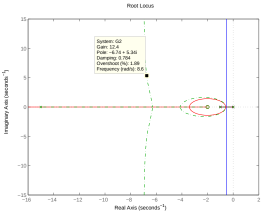
Example



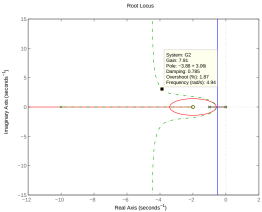
Plant model $P(s) = \frac{1}{s(s+1)}$



(a) $C(s) = s/2 + 1$



(b) $C(s) = \frac{s/2+1}{s/15+1}$

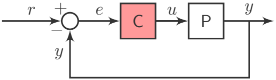


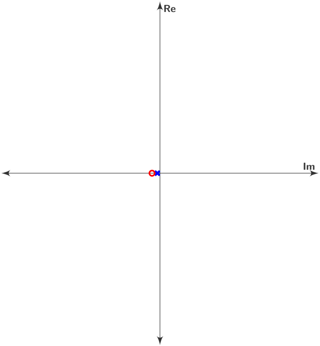
(c) $C(s) = \frac{s/2+1}{s/10+1}$

- Root Locus: $C(s)P(s) = -\frac{1}{K}$
- Location of z, p is based on trial and error
- Select desired closed-loop pole
  - Arbitrarily pick z, then use angle criterion to select p

Lag Compensator

Improves steady state performance





Compensator form

$$C(s) = K\frac{s+z}{s+p}$$

- $z > p > 0$ low frequency, near the origin
- z is close to p
- Root locus:

$$\frac{s+z}{s+p}P(s) = -\frac{1}{K}$$

- Boosts steady-state gain: $z/p > 1$.

Lag Compensator

Example

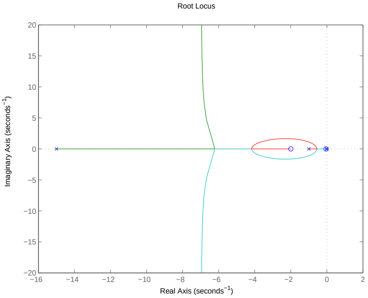
- Plant : $\frac{1}{s(s+1)}$, Lead Compensator: $\frac{K(s+2)}{s+15}$
- $K_v := \lim_{s\to 0} s\frac{K(s+2)}{s+15}\frac{1}{s(s+1)} = 90 \times 2/15 = 12.$
- Steady-state to ramp input $= 1/K_v = 1/12 = 0.0833$
- How to increase K_v? reduce e_{ss} to ramp

- Introduce a lag compensator: $\frac{s+0.05}{s+0.01}$
- $K_v := \lim_{s\to 0} s\frac{K(s+0.05)}{s+0.01}\frac{s+2}{s+15}\frac{1}{s(s+1)} = 5 \times 12 = 60$
- Steady-state to ramp input $= 1/K_v = 1/60 = 0.0166$

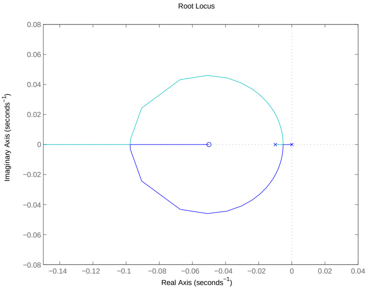
Lag compensators amplify gain at low frequency
Have no effect at high-frequency

Lag Compensator

Caution



(a) Effect of Lag Compensator



(b) Zoomed in

- Closed-loop poles are near the zero at -0.05
- Very slow decay rate.
- Proximity of poles $\implies$ low amplitude
- May affect settling time, especially for disturbance response

Put lag pole-zero at as high frequency possible, without affecting transients

Design Example

Control of a Small Airplane

Piper Dakota (from text book)

System

Transfer function from δ_e (elevator angle) to θ (pitch angle) is

$$P(s) = \frac{\theta(s)}{\delta_e(s)} = \frac{160(s+2.5)(s+0.7)}{(s^2+5s+40)(s^2+0.03s+0.06)}$$

Control Objective 1

Design an autopilot so that the step response to elevator input has $t_r < 1$ and $M_p < 10\% \implies \omega_n > 1.8$ rad/s and $\zeta > 0.6$ 2^{nd} order

Control of a Small Airplane

- Open Loop Poles: $-2.5 \pm 5.81j, -0.015 \pm 0.244j$ (stable)
- Open Loop Zeros: $-2.5, -0.7$ (no RHS zeros)

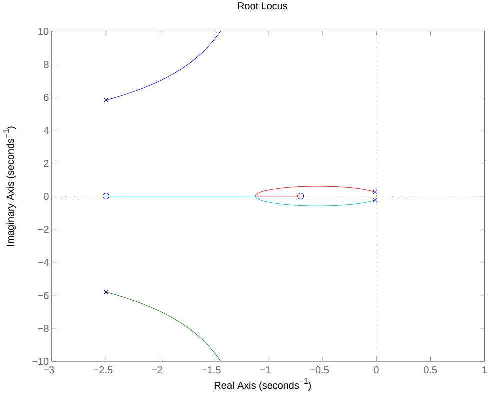
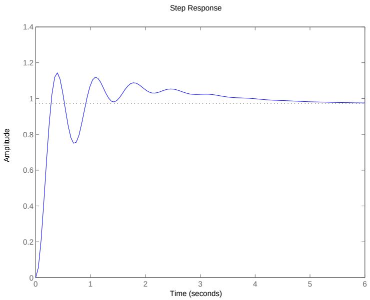


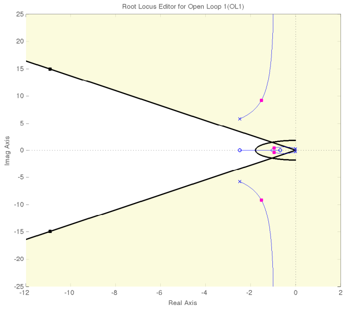
Figure : Root locus with proportional feedback

Control of a Small Airplane

Proportional Controller



(a) Step response with proportional



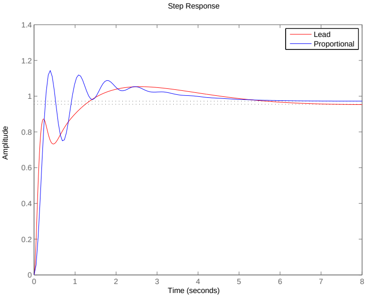
(b) Root locus with proportional

Not possible to satisfy ζ requirement with just proportional controller

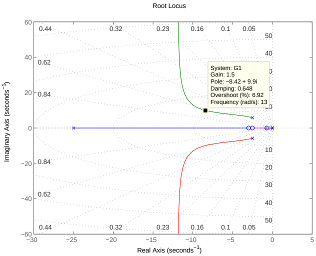
Control of a Small Airplane

Lead Compensator

After trial and error, choose $C(s) = K\frac{s+3}{s+25}$, with $K = 1.5$



(a) Step response with lead compensator



(b) Root locus with lead compensator

Has steady-state error ... have to fix this.

Control of a Small Airplane

Lead Compensator + Integral Control

Fix Steady-State Error

- introduce integral control

$$C(s) = KD_c(s)(1 + K_I/s)$$

- tune K_I to get desired behaviour
- study root locus w.r.t K_I

Characteristic Equation

$$1 + KD_c(s)P(s) + \frac{K_I}{s}KD_c(s)P(s) = 0$$

Write this in $L(s) = -\frac{1}{K_I}$ form

Control of a Small Airplane

Lead Compensator + Integral Control (contd.)

Characteristic Equation

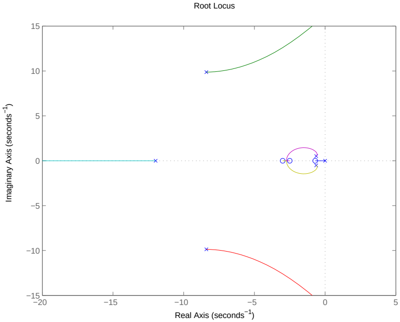
$$1 + KD_c(s)P(s) + \frac{K_I}{s}KD_c(s)P(s) = 0$$

Write this in $L(s) = -\frac{1}{K_I}$ form

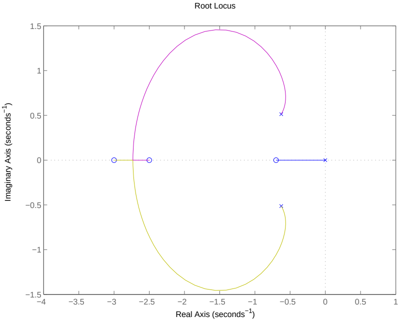
$$L(s) = \frac{1}{s}\frac{KD_cP}{1 + KD_cP}$$

Control of a Small Airplane

Lead Compensator + Integral Control (contd.)



(a) Root locus with PI



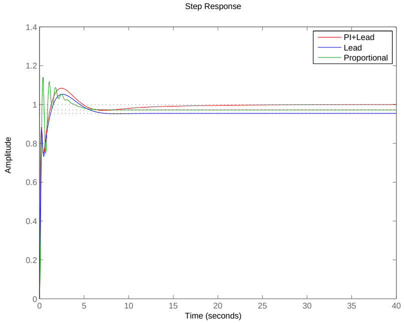
(b) Root locus with PI (zoomed)

- For $K_I > 0$, $\zeta \downarrow \implies M_p \uparrow$

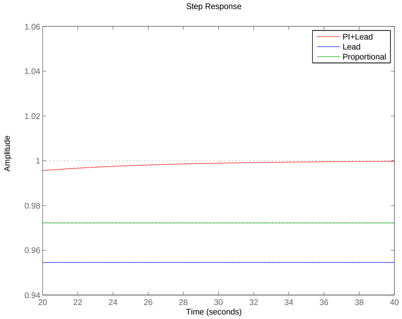
Control of a Small Airplane

Lead Compensator + Integral Control (contd.)

- Choose small value of $K_I = 0.15$
- Higher overshoot at the cost of zero steady-state error



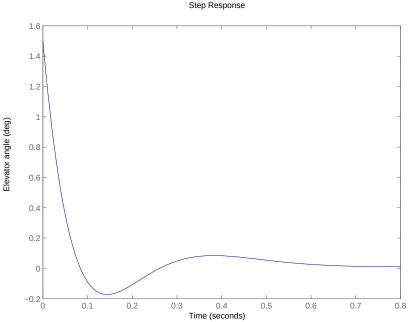
(a) Root locus with PI



(b) Root locus with PI (zoomed)

Control of a Small Airplane – Analysis

Control $u(t)$



- High frequency in $u(t)$ is undesirable rate limit & controller roll off
- Large values for $u(t)$ is undesirable saturation

Control of a Small Airplane – Analysis

How good is this controller?