AERO 422: Active Controls for Aerospace Vehicles

Basic Feedback Analysis & Design

Raktim Bhattacharya

Laboratory For Uncertainty Quantification
Aerospace Engineering, Texas A&M University.

Stabilizing Controller

Routh's Stability Criterion

Given characteristic equation

Factor out any roots at the origin and multiply by -1 if needed

$$D_G(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n,$$

How to determine stability? (without using MATLAB :)

- Necessary condition:
  - All coefficients of characteristic polynomial be positive, i.e. $a_i > 0$.
  - Any coefficient missing ($= 0$) or negative, then poles are outside LHP.
- Necessary and Sufficient condition:
  - System is stable iff all the elements in the first column of Routh array are positive

Routh's Stability Criterion

Routh Array

$$
\begin{array}{ll|c|cccc}
\text{Row} & n & s^n & 1 & a_2 & a_4 & \cdots \\
\text{Row} & n-1 & s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
\text{Row} & n-2 & s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
\text{Row} & n-3 & s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\text{Row} & 2 & s^2 & * & * & * & \\
\text{Row} & 1 & s^1 & * & * & & \\
\text{Row} & 0 & s^0 & * & & &
\end{array}
$$

where

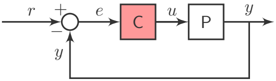
$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} \qquad b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} \qquad c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

There are two special cases! See web-appendix of textbook

Stabilizing Gain

Example 1 – One parameter



Given $P(s) = \frac{s+1}{s(s-1)(s+6)}$ and $C(s) = K$

design parameter

Characteristic Equation Numerator of $1 + PC$ No pole zero cancellation

$$s^3 + 5s^2 + (K-6)s + K = 0$$

Routh's Table

s^3	1	$K-6$
s^2	5	K
s^1	$\frac{4K-30}{5}$	
s^0	K	

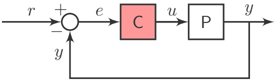
Stabilizing Gains

$$\frac{4K-30}{5} > 0, \; K > 0$$

$$K > 7.5, \; K > 0$$

Stabilizing Gain

Example 2 – Two parameters



Given $P(s) = \frac{1}{(s+1)(s+2)}$, $C(s) = K + \frac{K_I}{s}$

Characteristic Equation Numerator of $1 + PC$ No pole zero cancellation

$$s^3 + 3s^2 + (2+K)s + K_I = 0$$

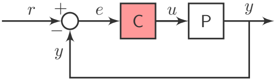
Routh's Table

$$\begin{array}{lll} s^3 & 1 & 2+K \\ s^2 & 3 & K_I \\ s^1 & \frac{6+3K-K_I}{3} & \\ s^0 & K_I & \end{array}$$

Stabilizing Gains $K > K_I/3 - 2,\ K_I > 0.$

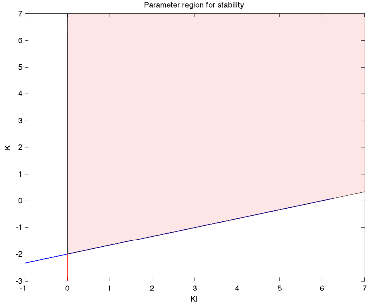
Stabilizing Gain

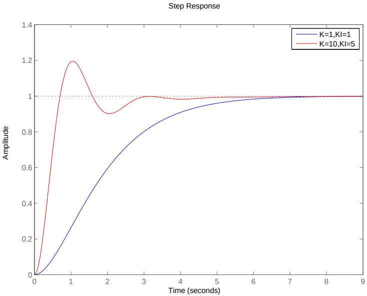
Example 1 – Two parameters (contd.)



Given $P(s) = \frac{1}{(s+1)(s+2)}$, $C(s) = K + \frac{K_I}{s}$

Stabilizing Gains $K > K_I/3 - 2,\ K_I > 0.$

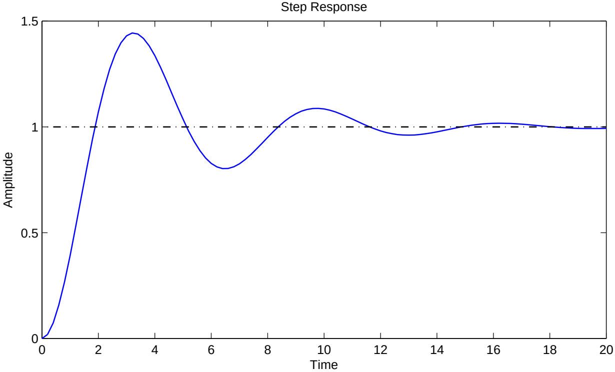




What about performance?

Step Response

Time Domain Performance Specification



Second Order System: poles $= \sigma \pm j\omega_d$, $\omega_n = \sqrt{\sigma^2 + \omega_d^2}, \zeta = \sigma/\omega_n$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_r = \frac{1.8}{\omega_n} \qquad t_s = \frac{4.6}{\sigma}$$

Step Response

Time Domain Performance Specification – Second Order Systems

Desired Location of Poles

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_r = \frac{1.8}{\omega_n} \qquad t_s = \frac{4.6}{\sigma}$$

$$\omega_n \geq 1.8/t_r \qquad \zeta \geq \zeta(M_p) \qquad \sigma \geq 4.6/t_s$$

- Adjust K, K_I to satisfy additional performance related constraints
- Controller gain tuning

Routh's Stability Criterion

Conditions that ensure $\mathbf{Re}\, p_i < -\alpha$, *for* $\alpha > 0$

Given polynomial

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

Modify Routh's stability criterion to ensure

$$\mathbf{Re}\, p_i < -\alpha, \text{ for } \alpha > 0.$$

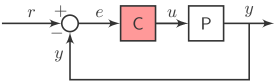
Replace $s := q - \alpha$ and substitute in polynomial to get

$$q^n + b_1 q^{n-1} + \cdots + b_{n-1}q + b_n = 0$$

Apply Routh's criterion to the polynomial in q

Routh's Stability Criterion

Example



- Given $P(s) = \frac{1}{s^2+4s+1}$ and controller $C(s) = K_1 + K_2/s$.
- Find range of values for K_1, K_2 such that all poles are left of $-\alpha$.

Characteristic equation:

$$s^3 + 4s^2 + (K_1 + 1)s + K_2$$

Substitute $s := q - \alpha$, with $\alpha = 1$

$$q^3 + q^2 + (K_1 - 4)q - K_1 + K_2 + 2$$

Apply Routh's criterion

Routh's Stability Criterion

Example (contd.)

Polynomial in q

$$q^3 + q^2 + (K_1 - 4)q - K_1 + K_2 + 2$$

Routh's Table

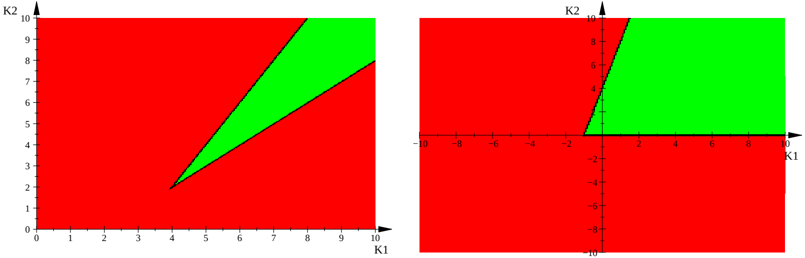
$$\begin{array}{c|cc} s^3 & 1 & K_1 - 4 \\ s^2 & 1 & K_2 - K_1 + 2 \\ s^1 & 2K_1 - K_2 - 6 & 0 \\ s^0 & K_2 - K_1 + 2 & 0 \end{array}$$

Inequalities

$$2K_1 - K_2 - 6 > 0, \; K_2 - K_1 + 2 > 0.$$

Routh's Stability Criterion

Example (contd.)



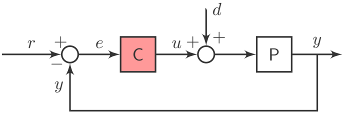
(a) $2K_1 - K_2 - 6 > 0, K_2 - K_1 + 2 > 0,$ Poles left of -1.

(b) $K_1 - K_2/4 + 1 > 0,\ K_2 > 0,$ Poles left of 0.

Benefits of Feedback

Benefits of Feedback

Disturbance Rejection



- Let $P(s) := \frac{A}{(s/p_1+1)(s/p_2+1)}$, and $C(s) := K$
- Total response

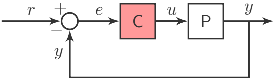
$$
\begin{aligned}
Y(s) &= \frac{PC}{1+PC}R(s) + \frac{P}{1+PC}D(s) \\
&= \frac{AK}{(s/p_1+1)(s/p_2+1)+AK}R(s) + \frac{A}{(s/p_1+1)(s/p_2+1)+AK}D(s)
\end{aligned}
$$

- Steady state value with feedback

$$\lim_{s\to 0} sY(s) = \frac{AK}{1+AK}\left(\lim_{s\to 0} sR(s)\right) + \frac{A}{1+AK}\left(\lim_{s\to 0} sD(s)\right)$$

Benefits of Feedback

Robustness to Plant Uncertainty



Transfer function from reference to output

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{AK}{(s/p_1+1)(s/p_2+1)+AK}$$

- Suppose $A \to A + \delta A$
- What is the effect on $T(s) = G_{yr}(s)$? steady state gain

Benefits of Feedback

Robustness to Plant Uncertainty (contd.)

$$T_{ss} = \frac{AK}{1+AK}$$

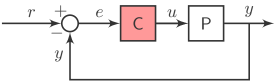
$$\begin{aligned}\delta T_{ss} &= \frac{dT_{ss}}{dA}\delta A \\ &= \frac{K}{(1+AK)^2}\delta A \\ &= \left(\frac{AK}{1+AK}\right)\left(\frac{1}{1+AK}\right)\frac{\delta A}{A}\end{aligned}$$

$$\implies \frac{\delta T_{ss}}{T_{ss}} = \frac{1}{1+AK}\frac{\delta A}{A}$$

System Type

System Type

Analysis of Steady State Error



$$E(s) = \frac{1}{1+PC}R(s)$$

$$\implies e_{ss} = \lim_{s\to 0} s \frac{1}{1+L(s)} \frac{1}{s^{k+1}}$$

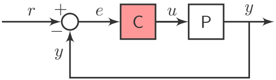
$$= \lim_{s\to 0} \frac{1}{1+L(s)} \frac{1}{s^{k}}$$

Investigate $e_{ss} \to 0$ for various values of k

Value of k	$r(t)$	System Type
0	$1(t)$	e_{ss} = constant $\implies$ Type 0
1	t	e_{ss} = constant $\implies$ Type I
2	$t^2/2!$	e_{ss} = constant $\implies$ Type II

Steady State Error

Type Zero



Type 0 System

- Constant steady state error to step reference $k = 0$.

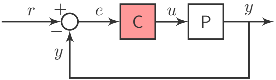
$$e_{ss} = \lim_{s\to 0} \frac{1}{1+L(s)}\frac{1}{s^k}$$
$$= \lim_{s\to 0} \frac{1}{1+L(s)} = \frac{1}{1+K_p}$$

Position Error Constant K_p

$$K_p = \lim_{s\to 0} L(s)$$

Steady State Error

Type One



Type 1 System

- Constant steady state error to ramp reference $k = 1$.

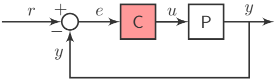
$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} \frac{1}{1+L(s)} \frac{1}{s^k} \\
&= \lim_{s\to 0} \frac{1}{1+L(s)} \frac{1}{s} = \lim_{s\to 0} \frac{1}{sL(s)} = \frac{1}{K_v}
\end{aligned}
$$

Velocity Error Constant K_v

$$K_v = \lim_{s\to 0} sL(s)$$

Steady State Error

Type Two



Type 2 System

- Constant steady state error to parabolic reference $k = 2$.

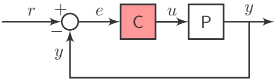
$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} \frac{1}{1+L(s)} \frac{1}{s^k} \\
&= \lim_{s\to 0} \frac{1}{1+L(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s^2 L(s)} = \frac{1}{K_a}
\end{aligned}
$$

Acceleration Error Constant K_a

$$K_a = \lim_{s\to 0} s^2 L(s)$$

Steady State Error

Summary



Various Constants

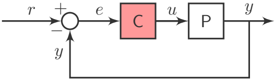
$$K_p = \lim_{s\to 0} L(s) \qquad K_v = \lim_{s\to 0} sL(s) \qquad K_a = \lim_{s\to 0} s^2 L(s)$$

Steady State Errors

	Step	Ramp	Parabola
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type I	0	$\frac{1}{K_v}$	∞
Type II	0	0	$\frac{1}{K_a}$

Steady State Error

Summary (contd.)



- Quickly identify ability to track polynomials
- Robustness property – higher type tracks lower order polynomials
- Can be extended to study G_{yd} and other transfer functions

Truxal's Formula

- Let $T(s) := G_{yr}(s)$ be given by

$$T(s) = K\frac{\Pi_{i=1}^{m}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)}.$$

- Most common case: e_{ss} to step is zero $\implies$ Type I system
- DC gain $\lim_{s\to 0} T(s) = 1$
- System error $E(s) = R(s)(1 - T(s))$
- System error due to ramp

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s(1 - T(s))\frac{1}{s^2} = \lim_{s\to 0} \frac{1 - T(s)}{s}$$

- Using L'Hopital's rule

$$e_{ss} = -\lim_{s\to 0} \frac{dT}{ds} = \frac{1}{K_v}$$ for type I systems

Truxal's Formula

Contd.

- Let $T(s) := G_{yr}(s)$ be given by

$$T(s) = K\frac{\Pi_{i=1}^{m}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)}.$$

- Using L'Hopital's rule

$$e_{ss} = -\lim_{s\to 0}\frac{dT}{ds} = \frac{1}{K_v} \text{ for type I systems}$$

- $\frac{1}{K_v}$ is related to the slope of $T(s)$ at origin

Truxal's Formula

Contd.

- Let $T(s) := G_{yr}(s)$ be given by

$$T(s) = K\frac{\Pi_{i=1}^{m}(s-z_i)}{\Pi_{i=1}^{n}(s-p_i)}.$$

- Rewrite

$$\begin{aligned} e_{ss} &= -\lim_{s\to 0}\frac{dT}{ds}\frac{1}{T} \qquad T(0)=1 \\ &= -\lim_{s\to 0}\frac{d}{ds}\log T(s) \\ &= -\lim_{s\to 0}\frac{d}{ds}\left(\log(K)+\sum_{i=1}^{m}\log(s-z_i)-\sum_{i=1}^{n}\log(s-p_i)\right) \end{aligned}$$

Truxal's Formula

$$\frac{1}{K_v} = \sum_{i=1}^{m}\frac{1}{z_i} - \sum_{i=1}^{n}\frac{1}{p_i}$$

Truxal's Formula

Design Implication

$$\frac{1}{K_v} = \sum_{i=1}^{m} \frac{1}{z_i} - \sum_{i=1}^{n} \frac{1}{p_i}$$

- Observe effect of pole/zero location on $1/K_v$
- Useful for design of dynamic compensators

Example

- Third order type I system has closed-loop poles $-2 \pm 2j, -0.1$.
- The system has one zero. Where should it be for $K_v = 10$?