Uncertainty & Robustness

Raktim Bhattacharya
Aerospace Engineering, Texas A&M University

Plant Uncertainty

Uncertainty in Models

No mathematical system can exactly model a physical system

- Must analyze how modeling errors affect performance of control system
- Basic technique is to characterize family of plants $\mathcal{P}$
- Design controller that achieves performance for all $P \in \mathcal{P}$

Kinds of uncertainty considered in this lecture

- Unstructured uncertainty – unmodeled dynamics, high frequency modes, etc
- Structured uncertainty – parametric variation or discrete set of plants

Unstructured Uncertainty in Models

We represent $\mathcal{P}$ as a disk about nominal $P_0(j\omega)$ with radius $W_2(j\omega)\Delta(j\omega)$

- $W_2(j\omega)$ is frequency dependent uncertainty profile – models gain uncertainty
- $\|\Delta(j\omega)\|_\infty < 1$, models phase uncertainty and also a scaling factor

Set $\mathcal{P}$ is therefore defined by

$$\mathcal{P} = P : P = (1 + \Delta W_2)P_0.$$

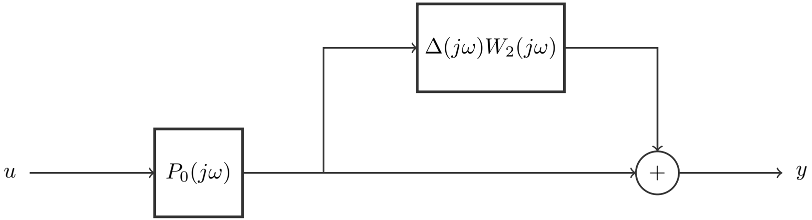
Assumptions

- $W_2(j\omega)$ is fixed stable transfer function
- $\Delta(j\omega)$ is variable stable transfer function with $\|\Delta(j\omega)\|_\infty < 1$
- No unstable poles of P_0 are cancelled in forming P – i.e. they both have same unstable poles

Unstructured Uncertainty in Models

Block Diagram Representation

Set $\mathcal{P} := P : P = (1 + \Delta W_2)P_0$ is equivalent to



Unstructured Uncertainty in Models

W_u represents percentage error

With $\mathcal{P}$ defined as

$$P = (1 + \Delta W_2)P_0,$$

Implies

$$\frac{P - P_0}{P_0} = \Delta W_2.$$

With $\|\Delta(j\omega)\|_\infty < 1$

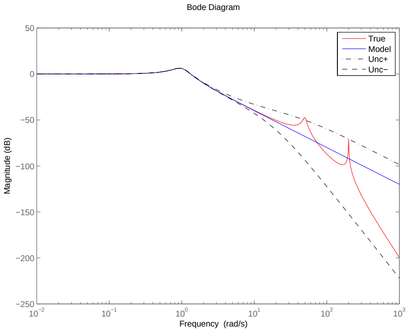
$$\left|\frac{P(j\omega) - P_0(j\omega)}{P_0(j\omega)}\right| \leq |W_2(j\omega)|, \ \forall\omega.$$

Inequality defines a disk in the complex plane, centered at 1.

Unstructured Uncertainty in Models

How to get W_u

- Obtain frequency-response from experiments
- Fit a nominal (average) plant
- Determine W_u so that $\mathcal{P}$ captures all the frequency responses



Unstructured Uncertainty in Models

Disk Uncertainty

Also known as multiplicative uncertainty

Advantages

- Simple – simplifies analysis
- Can say some fairly precise things

Disadvantages

- Conservative

Other forms of unstructured uncertainty

$$\mathcal{P} = \{P : P = P_0 + \Delta W_2\} \text{ additive uncertainty}$$

$$\mathcal{P} = \{P : P = \frac{P_0}{1 + \Delta W_2 P_0}\}$$

$$\mathcal{P} = \{P : P = \frac{P_0}{1 + \Delta W_2}\}$$

Robust Stability

Robust Stability

Definition

What is robust stability? Given $P \in \mathcal{P}$,

- consider some characteristics of the feedback system – e.g. internal stability
- controller $C(s)$ is robust with respect to this characteristic if this characteristic holds for all $P \in \mathcal{P}$.

Notion of robustness requires

- controller
- some characteristics – e.g. stability, performance
- a set $\mathcal{P}$

Another notion for robust stability can be considered

- Given $C(s)$, and $\mathcal{P}$ has an associated size
- What is the largest $\mathcal{P}$ that $C(s)$ can stabilize?

Robust Stability

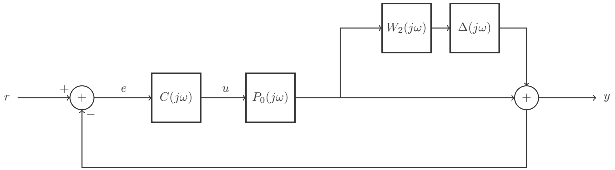
Condition for robust stability

Theorem For multiplicative uncertainty model, $C(s)$ provides robust stability iff

$$\|W_2(j\omega)T_0(j\omega)\|_\infty < 1, \text{ where } T_0 := \frac{P_0C}{1+P_0C}.$$

Proof: Based on Nyquist stability criterion, see pg. 53, Feedback Control Theory, Doyle, Francis, Tannenbaum.

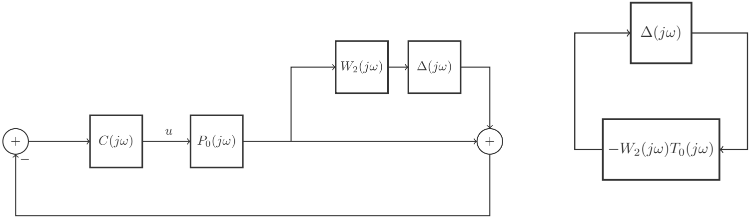
Another approach: Use small gain theorem.
Consider following interconnection:



Robust Stability

Condition for robust stability

Without exogenous signals, interconnection is

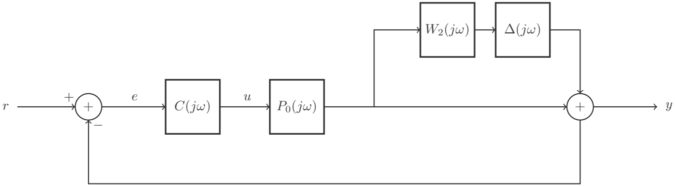


- Loop gain is $\Delta(j\omega)W_2(j\omega)T_0(j\omega)$
- Maximum loop gain is therefore $\|\Delta(j\omega)W_2(j\omega)T_0(j\omega)\|_\infty$
- $\|\Delta(j\omega)W_2(j\omega)T_0(j\omega)\|_\infty < 1$ for allowable $\Delta(j\omega)$ iff

$$\|W_2(j\omega)T_0(j\omega)\|_\infty < 1.$$

Robust Performance

Nominal Performance



- Consider tracking performance of controller $C(s)$

  a chosen characteristic of the control system

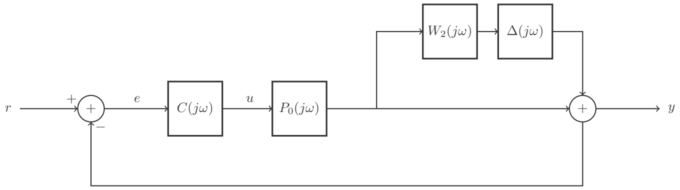
- Can be specified as

$$\|W_1(j\omega)S_0(j\omega)\|_\infty < 1,$$

where

$$S_0(j\omega) = \frac{1}{1 + P_0(j\omega)C(j\omega)}.$$

Robust Performance



Now consider perturbed plant $P = (1 + \Delta W_2)P_0$,

$$
\begin{aligned}
S := \frac{1}{1+PC} &= \frac{1}{1+(1+\Delta W_2)P_0C} \\
&= \frac{1}{1+P_0C+\Delta W_2 P_0 C} = \frac{1}{1+P_0C}\,\frac{1}{1+\Delta W_2 \frac{P_0C}{1+P_0C}} \\
&= \frac{S_0}{1+\Delta W_2 T_0}.
\end{aligned}
$$

Robust Performance

Condition

For robust performance, we require

- Robust stability

$$\|W_2(j\omega)T_0(j\omega)\|_\infty < 1.$$

- Tracking performance

$$\|W_1(j\omega)S(j\omega)\|_\infty < 1,$$

or

$$\left\| \frac{W_1(j\omega)S_0(j\omega)}{1 + \Delta(j\omega)W_2(j\omega)T_0(j\omega)} \right\|_\infty < 1, \; \forall\Delta.$$

Robust Performance

Condition (contd.)

Theorem A necessary and sufficient condition for robust performance is

$$\| \, |W_1(j\omega)S_0(j\omega)| + |W_2(j\omega)T_0(j\omega)| \, \|_\infty < 1.$$

Proof: see pg. 54, Feedback Control Theory, Doyle, Francis, Tannenbaum.

MIMO Systems

Robust Stability

Multiplicative uncertainty

For MIMO systems, multiplicative uncertainty is modeled as

$$\mathcal{P} := (I + W_1 \Delta W_2) P_0,$$

where $\Delta, W_1, W_2 \in \mathcal{RH}_\infty$.

Lemma: The feedback system with

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \; K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix},$$

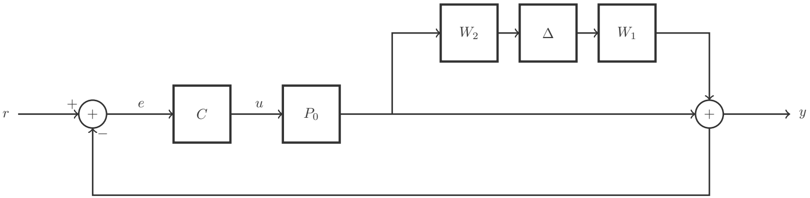
is well poised iff

$$\begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}, \text{ is invertible.}$$

Well-posed $\iff$ Feedback system is physically realizable.

Unstructured Robust Stability

Multiplicative Uncertainty



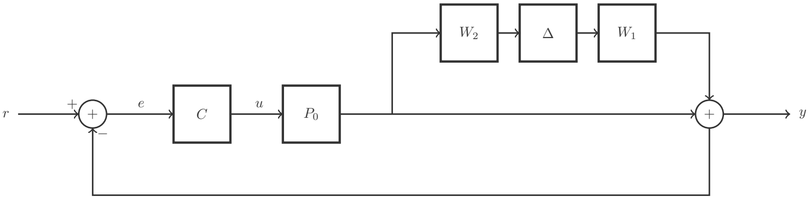
Theorem Closed-loop system is well-poised and internally stable for all $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < 1$ iff

$$\|W_2(j\omega)T_0(j\omega)W_1(j\omega)\|_\infty \leq 1.$$

Proof: Pg 223, Robust and Optimal Control, Kemin Zhou, John C. Doyle, Keith Glover.

Unstructured Robust Performance

$\mathcal{H}_\infty$ Performance

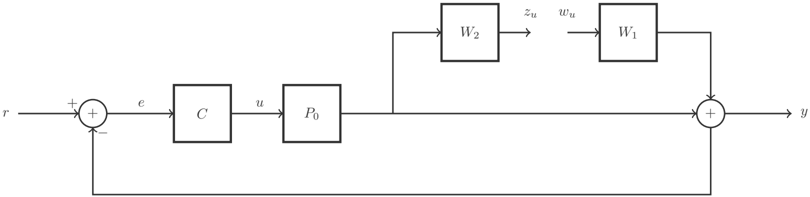


Keep worst-case energy of $z(t)$ small, over all $w(t)$ of unit energy

$$\|G_{w\to z}\|_\infty \leq 1, \; \forall P \in \mathcal{P}$$

Unstructured Robust Performance

$\mathcal{H}_\infty$ Performance (contd.)

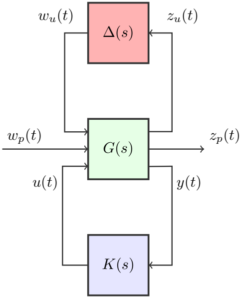


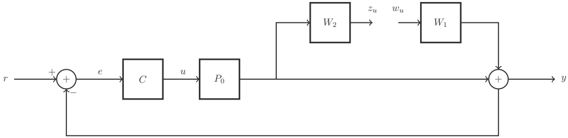
Keep worst-case energy of $z(t)$ small, over all $w(t)$ of unit energy

$$\|G_{w\to z}\|_\infty \leq 1, \; \forall P \in \mathcal{P}$$

Unstructured Robust Performance

$\mathcal{H}_\infty$ Performance (contd.)





Keep worst-case energy of $z(t) := \begin{bmatrix} z_u \\ z_p \end{bmatrix}$ small,

over all $w(t) := \begin{bmatrix} w_u \\ w_p \end{bmatrix}$ of unit energy

$$\|G_{w\to z}\|_\infty \leq 1, \; \forall P \in \mathcal{P}$$

$\Delta(s)$ is a full-block transfer matrix of size $n_{w_u} \times n_{z_u}$

Linear Fractional Transformation

Linear Fractional Transformation

- Let $F : \mathbb{C} \to \mathbb{C}$ of the form

$$F(s) = \frac{a + bs}{c + ds},$$

  where $a, b, c, d \in \mathbb{C}$ is called a linear fractional transformation (LFT).

- In particular, when $c \neq 0$, $F(s)$ can be written as

$$F(s) = \alpha + \beta s(1 - \gamma s)^{-1},$$

  for some $\alpha, \beta, \gamma \in \mathbb{C}$.

- Can be generalized to complex matrices.

Linear Fractional Transformation

Matrix Generalization

Let M be a complex matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)}.$$

Lower LFT with Δ_l is a map

$$\mathcal{F}_l(M,\Delta_l) := M_{11} + M_{12}\Delta_l\left(I - M_{22}\Delta_l\right)^{-1} M_{21},$$

provided $\left(I - M_{22}\Delta_l\right)^{-1}$ exists.

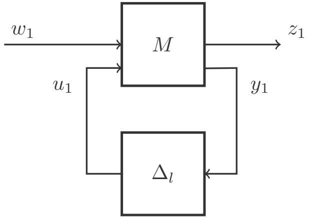
Upper LFT with Δ_u is a map

$$\mathcal{F}_u(M,\Delta_u) := M_{22} + M_{21}\Delta_u\left(I - M_{11}\Delta_u\right)^{-1} M_{12},$$

provided $\left(I - M_{11}\Delta_u\right)^{-1}$ exists.

Block Diagram Representation

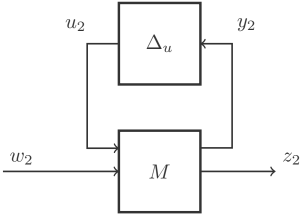
$$\mathcal{F}_l(M, \Delta_l) = G_{w_1 \to z_1}$$



$$\begin{bmatrix} z_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ u_1 \end{bmatrix}$$

$$u_1 = \Delta_l \, y_1$$

$$\mathcal{F}_u(M, \Delta_u) = G_{w_2 \to z_2}$$



$$\begin{bmatrix} y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u_2 \\ w_2 \end{bmatrix}$$

$$u_2 = \Delta_u \, y_2$$

Properties of LFT

Property 1 For compatible matrices A, B, C, D, Q, with C invertible,

$$\mathcal{F}_l(M, Q) = (A + BQ)(C + DQ)^{-1},$$
$$\mathcal{F}_l(N, Q) = (C + QD)^{-1}(A + QB),$$

where

$$M = \begin{bmatrix} AC^{-1} & B - AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix},$$
$$N = \begin{bmatrix} C^{-1}A & C^{-1} \\ B - DC^{-1}A & -DC^{-1} \end{bmatrix}.$$

- The converse also holds if M satisfies some conditions
- A, B, C, D are derived from M

Properties of LFT

Property 2: Equivalence

$$\mathcal{F}_u(M,\Delta) = \mathcal{F}_l(N,\Delta)$$

with

$$N = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M22 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}.$$

Property 3: Inverse Suppose $\mathcal{F}_u(M,\Delta)$ is square and well-defined and M_{22} is nonsingular. Then the inverse of $\mathcal{F}_u(M,\Delta)$ exists and is also an LFT w.r.t Δ, i.e.

$$\mathcal{F}_u(M,\Delta)^{-1} = \mathcal{F}_u(N,\Delta),$$

where

$$N = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{bmatrix}.$$

Properties of LFT

Addition

$$\mathcal{F}_u(M, \Delta_1) + \mathcal{F}_u(Q, \Delta_2) = \mathcal{F}_u(N, \Delta),$$

where

$$N = \begin{bmatrix} M_{11} & 0 & M_{12} \\ 0 & Q_{11} & Q_{12} \\ M_{21} & Q_{21} & M_{22} + Q_{22} \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

Product

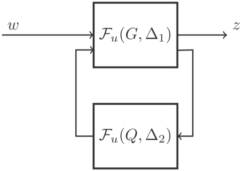
$$\mathcal{F}_u(M, \Delta_1)\,\mathcal{F}_u(Q, \Delta_2) = \mathcal{F}_u(N, \Delta),$$

where

$$N = \begin{bmatrix} M_{11} & M_{12}Q_{21} & M_{12}Q_{22} \\ 0 & M_{11} & Q_{12} \\ M_{21} & M_{22}Q_{21} & M_{22}Q_{22} \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

Properties of LFT

LFT of LFTs



$$
\begin{aligned}
&\mathcal{F}_l\left(\mathcal{F}_u\left(G,\Delta_1\right),\mathcal{F}_u\left(Q,\Delta_2\right)\right),\\
=&\mathcal{F}_u\left(\mathcal{F}_l\left(G,\mathcal{F}_u\left(Q,\Delta_2\right)\right),\Delta_1\right),\\
=&\mathcal{F}_u\left(N,\Delta\right).
\end{aligned}
$$

$$
G=\begin{bmatrix} A & B_1 & B_2\\ C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{bmatrix}
$$

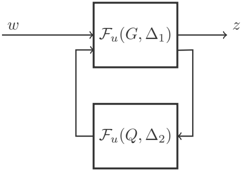
$$
N=\begin{bmatrix} A+B_2Q_{22}L_1C_2 & B_2L_2Q_{21} & B_1+B_2Q_{22}L_1D_{21}\\ Q_{12}L_1C_2 & Q_{11}+Q_{12}L_1D_{22}Q_{21} & Q_{12}L_1D_{21}\\ C_1+D_{12}L_2Q_{22}C_2 & D_{12}L_2Q_{21} & D_{11}+D_{12}Q_{22}L_1D_{21}\end{bmatrix},
$$

where

$$
L_1=(I-D_{22}Q_{22})^{-1}, L_2=(I-Q_{22}D_{22})^{-1}, \text{ and } \Delta=\begin{bmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{bmatrix}.
$$

Properties of LFT

LFT of LFTs



$$
\begin{aligned}
&\mathcal{F}_l\left(\mathcal{F}_u\left(G,\Delta_1\right),\mathcal{F}_u\left(Q,\Delta_2\right)\right),\\
=&\mathcal{F}_u\left(\mathcal{F}_l\left(G,\mathcal{F}_u\left(Q,\Delta_2\right)\right),\Delta_1\right),\\
=&\mathcal{F}_u\left(N,\Delta\right).
\end{aligned}
$$

If open-loop system parameters are LFTs of some variable,

- closed-loop system parameters are LFTs of the same variable
- useful for perturbation analysis and building interconnections

Examples of LFTs

Polynomials

Let

$$p(\delta) = a_0 + a_1\delta + a_2\delta^2 + \cdots + a_n\delta^n.$$

It can be verified that

$$p(\delta) = \mathcal{F}_l\left(M, \delta I_n\right),$$

with

$$M = \left[\begin{array}{c|cccc} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ \hline 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right].$$

- every polynomial can be written in LFT form
- can be extended to multi-variate polynomials

Examples of LFTs

State-Space Realizations

Dynamical system

$$\dot{x} = Ax + bu, y = Cx + Du,$$

has transfer function

$$G(s) = D + C(sI - A)^{-1}B = \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{s}I\right).$$

Compare with

$$\mathcal{F}_u(M, \Delta_u) := M_{22} + M_{21}\Delta_u\left(I - M_{11}\Delta_u\right)^{-1}M_{12}.$$

Parametric Uncertainty

Mass Spring Damper System

Consider the dynamical system

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}.$$

The system has parametric uncertainty in m, c, k modeled as multiplicative uncertainty

$$m = m_0(1 + 0.1\delta_m), \qquad 10\% \text{ off nominal}$$
$$c = c_0(1 + 0.2\delta_c), \qquad 20\% \text{ off nominal}$$
$$k = k_0(1 + 0.3\delta_k), \qquad 30\% \text{ off nominal} .$$

Write $\frac{1}{m}$ as an LFT

$$\frac{1}{m} = \frac{1}{m_0(1 + 0.1\delta_m)} = \frac{1}{m_0} - \frac{0.1}{m_0}\delta_m(1 + 0.1\delta_m)^{-1} = \mathcal{F}_l\left(M, \delta_m\right),$$

Parametric Uncertainty

Mass Spring Damper System

Write $\frac{1}{m}$ as an LFT

$$\frac{1}{m} = \frac{1}{m_0(1+0.1\delta_m)} = \frac{1}{m_0} - \frac{0.1}{m_0}\delta_m(1+0.1\delta_m)^{-1} = \mathcal{F}_l\left(M_1, \delta_m\right),$$

where

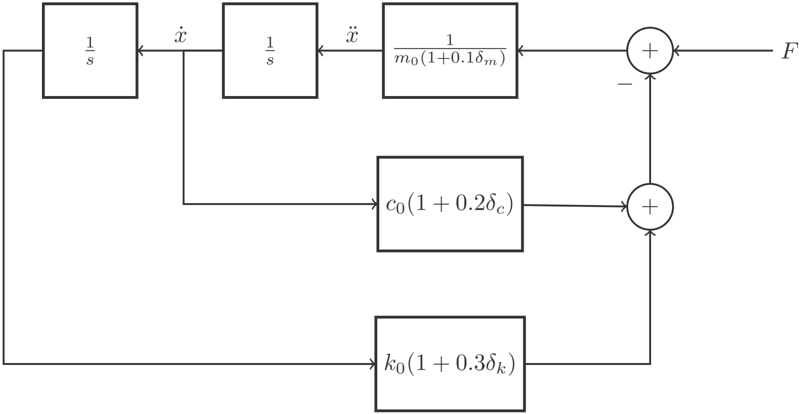
$$M_1 = \begin{bmatrix} \frac{1}{m_0} & -\frac{0.1}{m_0} \\ 1 & -0.1 \end{bmatrix}.$$

Rest c, k are linear in δ_c, δ_m.

Parametric Uncertainty

Mass Spring Damper System

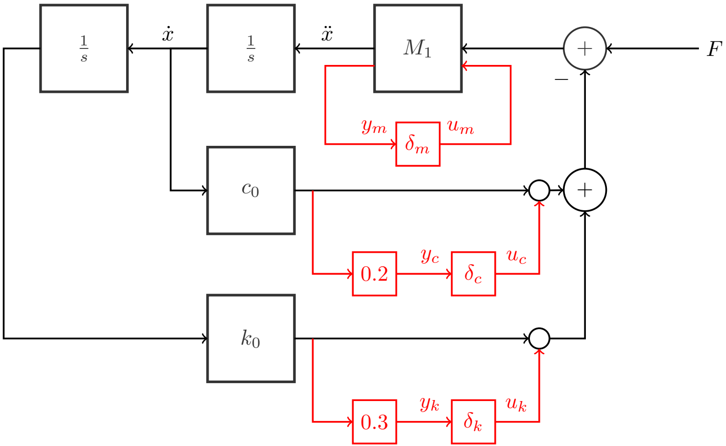
Block Diagram Representation



Parametric Uncertainty

Mass Spring Damper System

Representation with △ Blocks



Parametric Uncertainty

Mass Spring Damper System

In state-space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_k \\ y_c \\ y_m \end{bmatrix} = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & -\frac{1}{m_0} & -\frac{1}{m_0} & -\frac{1}{m_0} & -\frac{0.1}{m_0} \\ \hline 0.3k_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2c_0 & 0 & 0 & 0 & 0 \\ -k_0 & -c_0 & 1 & -1 & -1 & -0.1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ F \\ u_k \\ u_c \\ u_m \end{bmatrix},$$

$$\begin{bmatrix} u_k \\ u_c \\ u_m \end{bmatrix} = \Delta \begin{bmatrix} y_k \\ y_c \\ y_m \end{bmatrix}, \quad \text{and } \Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}.$$

Parametric Uncertainty

Mass Spring Damper System

In LFT form, we can write this as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathcal{F}_l\left(M, \Delta\right) \begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix},$$

where

$$M = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & \frac{1}{m_0} & -\frac{1}{m_0} & -\frac{1}{m_0} & -\frac{0.1}{m_0} \\ \hline 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2c_0 & 0 & 0 & 0 & 0 \\ -k_0 & -c_0 & 1 & -1 & -1 & -0.1 \end{array}\right].$$

Parametric Uncertainty

General Affine State-Space Uncertainty

Consider a linear system parameterized by k parameters

$$A := A_0 + \sum_{i=1}^{k} \delta_i A_i, \qquad B := B_0 + \sum_{i=1}^{k} \delta_i B_i$$

$$C := C_0 + \sum_{i=1}^{k} \delta_i C_i, \qquad D := D_0 + \sum_{i=1}^{k} \delta_i D_i$$

where $\delta_i \in [-1, 1]$ represents the i^{th} uncertainty.
Transfer matrix is therefore

$$G_\delta(s) = \mathcal{F}_u\left(M_\delta, \frac{1}{s}I\right),$$

where

$$M_\delta := \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$ Write M_δ in LFT form

Parametric Uncertainty

General Affine State-Space Uncertainty (contd.)

Define matrix P_i as

$$P_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i. \end{bmatrix}$$

Let q_i be the rank of P_i.
Then

$$P_i = \begin{bmatrix} L_i \\ W_i \end{bmatrix} \begin{bmatrix} R_i \\ Z_i \end{bmatrix}^*,$$

where $L_i \in \mathbb{R}^{n\times q_i}, W_i \in \mathbb{R}^{n_y\times q_i}, R_i \in \mathbb{R}^{n\times q_i}, W_i \in \mathbb{R}^{n_u\times q_i}$.
Therefore

$$\delta_i P_i = \begin{bmatrix} L_i \\ W_i \end{bmatrix} (\delta_i I) \begin{bmatrix} R_i \\ Z_i \end{bmatrix}^*.$$

Parametric Uncertainty

General Affine State-Space Uncertainty (contd.)

M_δ can be written as

$$M_\delta = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{M_{11}} + \underbrace{\begin{bmatrix} L_1 & L_2 & \cdots & L_k \\ W_1 & W_2 & \cdots & W_k \end{bmatrix}}_{M_{12}} \underbrace{\begin{bmatrix} \delta_1 I_{q_1} & & \\ & \ddots & \\ & & \delta_k I_{q_k} \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} R_1^* & Z_1^* \\ \vdots & \vdots \\ R_k^* & Z_k^* \end{bmatrix}}_{M_{21}}$$

or

$$M_\delta = \mathcal{F}_l\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix}, \Delta\right).$$

Therefore

$$G_\delta(s) = \mathcal{F}_u\left(\mathcal{F}_l\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix}, \Delta\right), \frac{1}{s}I\right).$$