State Feedback Control Synthesis - Part II

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$\mathcal{H}_\infty$ Optimal Controller

Motivation

- $\mathcal{H}_2$ Optimal Control
  - disturbance error reduction
  - sensor noise error reduction
- $\mathcal{H}_\infty$ Optimal Control
  - disturbance error reduction
  - sensor noise error reduction
  - tolerant to uncertainties – easier to formulate in $\mathcal{RH}_\infty$ than $\mathcal{RH}_2$

	$\|u\|_2$	$\|u\|_\infty$	$\mathbf{pow}(u)$
$\|y\|_2$	$\|\hat{G}(j\omega)\|_\infty$	∞	∞
$\|y\|_\infty$	$\|\hat{G}(j\omega)\|_2$	$\|G(t)\|_1$	∞
$\mathbf{pow}(y)$	0	$\leq \|\hat{G}(j\omega)\|_\infty$	$\|\hat{G}(j\omega)\|_\infty$

∞-norm of system is pretty useful

Kalman-Yakubovich-Popov (KYP) Lemma

Lemma: Suppose $\hat{G}(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then the following are equivalent conditions.

1. The matrix A is Hurwitz and

$$\|\hat{G}\|_\infty < 1.$$

2. There exists a matrix $X > 0$ such that

$$\begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} < 0.$$

- Very useful – relates transfer matrix (frequency domain) inequality to state space conditions
- Convenient way to evaluate $\mathcal{H}_\infty$ norm of transfer matrix

Full State-Feedback $\mathcal{H}_\infty$ Control

One of three formulations

Given system

$$\dot{x} = Ax + B_u u + B_w w,$$
$$z = Cx + D_u u + D_w w.$$

Theorem Controller $u = Kx$ internally stabilizes and minimizes $\|G_{w\to z}\|_\infty$ iff there exists W, and $X > 0$ such that following optimization problem has solution (A, B_u) stabilizable

$$\min_{X,W} \gamma$$

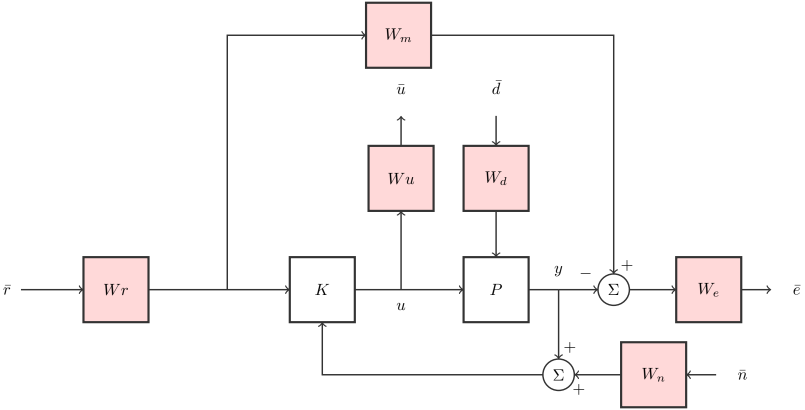
subject to

$$X > 0, \begin{bmatrix} (AX + B_u W) + (*)^T & B_w & (CX + D_u W)^T \\ B_w^T & -\gamma I & D_w^T \\ (CX + D_u W) & D_w & -\gamma I \end{bmatrix} < 0,$$

with $K = WX^{-1}$.

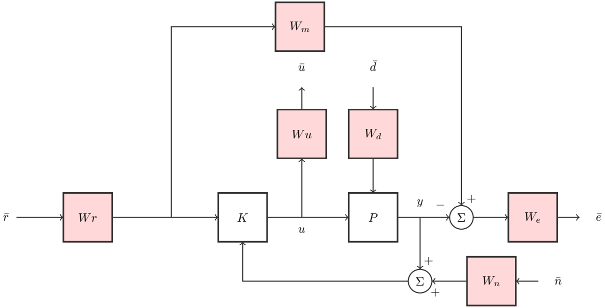
Weighted Performance

For both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control



Standard interconnection

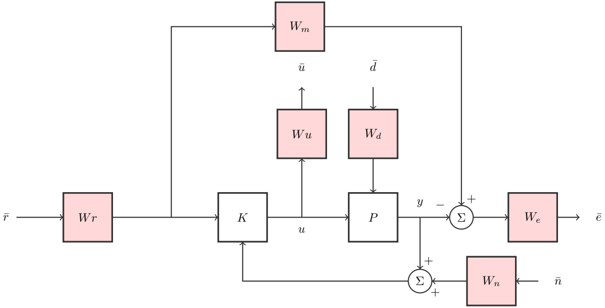
Frequency Dependent Weights



- Some signals may be more important than others
- Signals may not be measured in the same metric
- May be interested in keeping signals small in certain frequency range

Frequency Dependent Weights

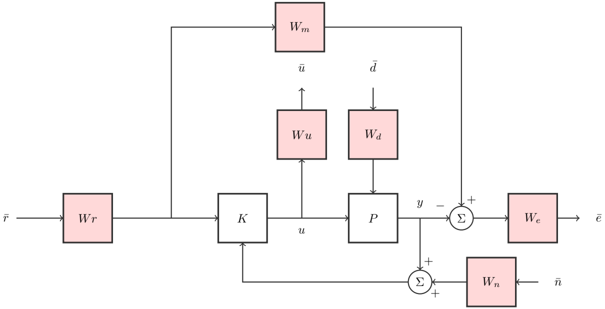
W_r, W_d, W_n



- W_r: specifies frequency content of $r(t)$ – Pilot models, etc.
- W_d: specifies frequency content of $d(t)$ – gust models, road vibration, etc.
- W_n: specifies frequency content of sensor noise – comes from manufacturer.

Frequency Dependent Weights

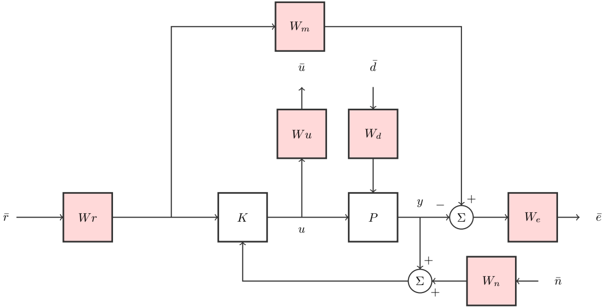
W_u



- W_u: defines the reciprocal of desired frequency content of $u(t)$
- Can be used to
  - include control magnitude, rate constraints
  - specify desired controller roll off – not excite high-frequency uncertain modes

Frequency Dependent Weights

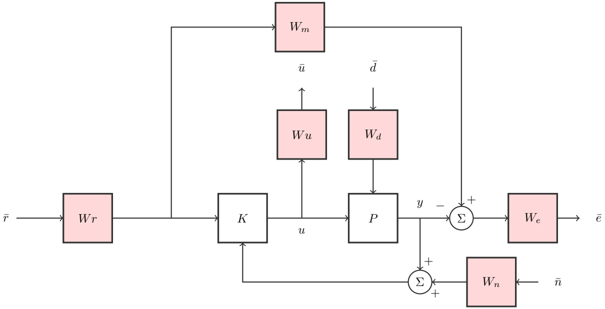
W_e



- W_e: defines the reciprocal of desired error at each frequency

Frequency Dependent Weights

W_m



- W_m: Defines the model for model-matching formulation
- Desired response to $r(t)$ is given by respond of model W_m
- E.g. second order response – can relate to rise time, overshoot, settling time

$\mathcal{H}_\infty$ Loopshaping – $P(j\omega)C(j\omega)$

Define desired loop shape using weights

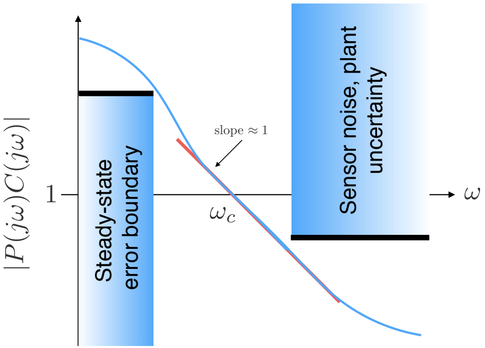
Develop conditions on the Bode plot of the open loop transfer function

- Sensitivity $\frac{1}{1+PC}$
- Steady-state errors: slope and magnitude at $\lim_\omega \to 0$
- Robust to sensor noise
- Disturbance rejection
- Controller roll off $\implies$ not excite high-frequency modes of plant
- Robust to plant uncertainty

Look at Bode plot of $L(j\omega) := P(j\omega)C(jw)$

Frequency Domain Specifications

Constraints on the shape of $L(j\omega)$

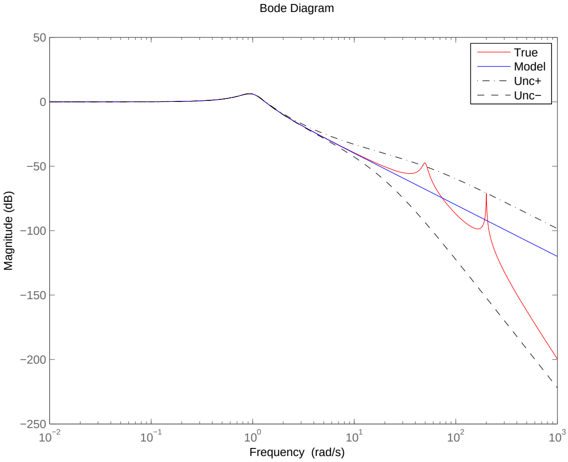


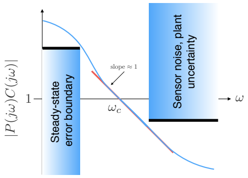
- Choose $C(j\omega)$ to ensure $|L(j\omega)|$ does not violate the constraints
- Slope ≈ -1 at ω_c ensures $PM \approx 90^\circ$

stable if $PM > 0 \implies \underline{/PC} > -180^\circ$

Plant Uncertainty

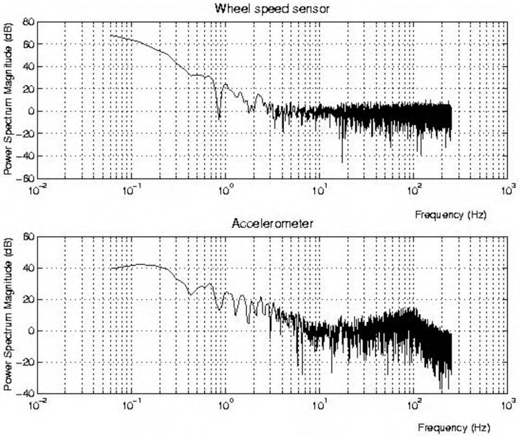
$$P(j\omega) = P_0(j\omega)(1 + \Delta P(j\omega))$$

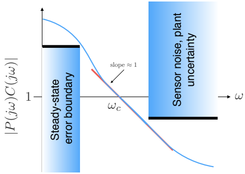




Sensor Characteristics

Noise spectrum

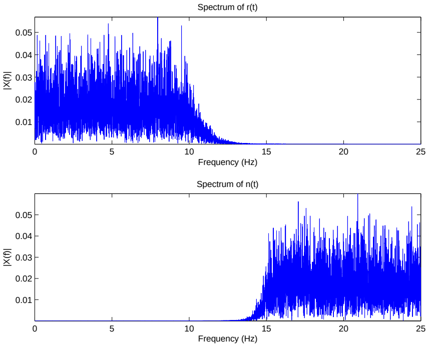


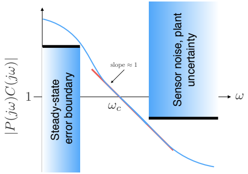


$$G_{yn} = -\frac{PC}{1+PC}$$

Reference Tracking

Bandlimited else conflicts with noise rejection



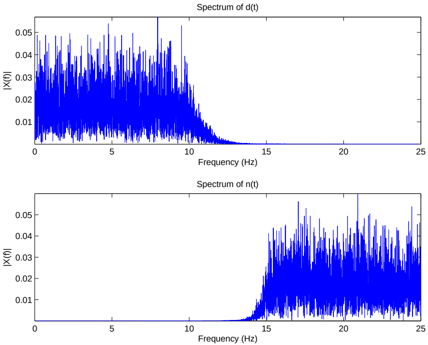


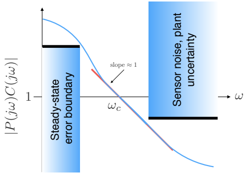
$$G_{yr} = \frac{PC}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$

Disturbance Rejecton

Bandlimited *else conflicts with noise rejection*





$$G_{yd} = \frac{P}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$

State Observer

Full Order Observer

Consider the system

$$\dot{x} = Ax + Bu,\ y = Cx.$$

Full order state observer takes the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y),$$

where

- $\hat{x}$ is state observation vector
- L is the observer gain

Define

$$e = x - \hat{x},$$

therefore

$$\dot{e} = (A + LC)e.$$

Full Order Observer

Design Problem

Problem Design L so that

$$\lim_{t\to\infty} e(t) := x(t) - \hat{x}(t) = 0.$$

Solution 1 It has a solution iff $\exists\, P > 0$ and W such that

$$PA + A^T P + WC + C^T W < 0,$$

and observer gain is

$$L = P^{-1}W.$$

Full-order state observer design is dual to state-feedback controller design.

$\mathcal{H}_\infty/\mathcal{H}_2$ Observer

Problem Setup

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_w w, \\ y &= C_y x + D_u u + D_w w, \\ z &= C_z x. \end{aligned}$$

Full-Order State Observer Design L for

$$\dot{\hat{x}} = (A + LC_y)\hat{x} - Ly + (B_u + LD_u)u,$$

such that w has little effect on

$$\hat{z} = C_z \hat{x},$$

which is estimate of interested output.

Problem Setup

contd.

Define

$$e = x - \hat{x}, \tilde{z} = z - \bar{z}.$$

The observation error system is therefore,

$$\begin{aligned} \dot{e} &= (A + LC_y)e + (B_w + LD_w)w, \\ \tilde{z} &= C_z e. \end{aligned}$$

The transfer function $\hat{G}_{w\to\tilde{z}}$ is therefore,

$$\hat{G}_{w\to\tilde{z}} = C_z(sI - A - LC_y)^{-1}(B_w + LD_w),$$

which is independent of u.

Problem Setup

contd.

With

$$\hat{G}_{w\to\tilde{z}} = C_z(sI - A - LC_y)^{-1}(B_w + LD_w),$$

$\mathcal{H}_2$ Observer

$$\min_L \gamma,$$
$$\|\hat{G}_{w\to\tilde{z}}\|_2 < \gamma.$$

$\mathcal{H}_\infty$ Observer

$$\min_L \gamma,$$
$$\|\hat{G}_{w\to\tilde{z}}\|_\infty < \gamma.$$

$\mathcal{H}_\infty$ State Observer Design

The optimization problem

$$\min_{L} \gamma,$$
$$\|\hat{G}_{w\to\tilde{z}}\|_\infty < \gamma,$$

has solution iff $\exists\, W$ and $P > 0$ such that

$$\min_{W,P} \gamma$$

such that

$$\begin{bmatrix} A^TP + C_y^TW^T + (*)^T & PB_w + WD_w & C_z^T \\ (PB_w + WD_w)^T & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} < 0.$$

$\mathcal{H}_2$ State Observer Design

The optimization problem

$$\min_{L} \gamma,$$
$$\|\hat{G}_{w\rightarrow\tilde{z}}\|_2 < \gamma,$$

has solution iff $\exists\, W, Q > 0$, and $X > 0$ such that

$$\min_{W,Q,X} \gamma$$

such that

$$\mathbf{tr} Q < \gamma$$
$$\begin{bmatrix} XA + WC_y + (*)^T & XB_w + WD_w \\ (XB_w + WD_w)^T & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -Q & C_z \\ C_z^T & -X \end{bmatrix} < 0.$$

$\mathcal{H}_\infty/\mathcal{H}_2$ Filtering

$\mathcal{H}_\infty$ Filtering

Problem Formulation

Here we consider the dynamical system

$$
\begin{aligned}
\dot{x} &= Ax + Bw,\ x(0) = x_0,\\
y &= Cx + Dw,\\
z &= Lx.
\end{aligned}
$$

- Filtering is state-estimation for stochastic systems
- In this framework w need not be stochastic
- Design objective same: eliminate the effect of disturbance from estimate of z as much as possible
- System matrix A is closed-loop dynamics and stable

$\mathcal{H}_\infty$ Filtering

Problem Formulation (contd.)

System Dynamics

$$\dot{x} = Ax + Bw,\ x(0) = x_0,$$
$$y = Cx + Dw,$$
$$z = Lx.$$

Unknown Filter Dynamics

$$\dot{x}_F = A_F x_F + B_F y,\ x_F(0) = x_{F_0},$$
$$\hat{z} = C_F x_F + D_F y.$$

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix},\ \tilde{z} = z - \hat{z},$$

dynamics is

$$\dot{x}_e = \tilde{A}x_e + \tilde{B}w,\ x_e(0) = x_{e0}$$
$$\tilde{z} = \tilde{C}x_e + \tilde{D}w.$$

$\mathcal{H}_\infty$ Filtering

Problem Formulation (contd.)

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix},\ \tilde{z} = z - \hat{z},$$

dynamics is

$$\begin{aligned} \dot{x}_e &= \tilde{A}x_e + \tilde{B}w,\ x_e(0) = x_{e0} \\ \tilde{z} &= \tilde{C}x_e + \tilde{D}w, \end{aligned}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} L - D_F C & -C_F \end{bmatrix}, \qquad \tilde{D} = -D_F D.$$

$\mathcal{H}_\infty$ Filtering

Synthesis

Optimization Problem

$$\min_{A_F,B_F,C_F,D_F} \gamma, \ \|\hat{G}_{w\to\tilde{z}}\|_\infty < \gamma.$$

or

$$\min_{R,X,M,N,D_F} \gamma$$

subject to

$$\begin{bmatrix} RA + A^TR + ZC + C^TZ^T & * & * & * \\ M^T + ZC + XA & M^T + M & * & * \\ B^TR + D^TZ^T & B^TX + D^TZ^T & \gamma I & * \\ L - D_FC & -N & -D_FF & \gamma I \end{bmatrix} < 0,$$

and $X > 0, R - X > 0$.

$\mathcal{H}_\infty$ Filtering

Synthesis

Optimization Problem

$$\min_{R,X,M,N,D_F} \gamma$$

subject to

$$\begin{bmatrix} RA + A^TR + ZC + C^TZ^T & * & * & * \\ M^T + ZC + XA & M^T + M & * & * \\ B^TR + D^TZ^T & B^TX + D^TZ^T & -\gamma I & * \\ L - D_FC & -N & -D_FD & -\gamma I \end{bmatrix} < 0,$$

and $X > 0, R - X > 0.$

Filter Dynamics

$$A_F = X^{-1}M,\ B_F = X^{-1}Z,\ C_F = N.$$

Proof:

A linear matrix inequality approach to robust $\mathcal{H}_\infty$ filtering – Huaizhong Li, Minyue Fu.

$\mathcal{H}_2$ Filtering

Problem Formulation

System Dynamics

$$
\begin{aligned}
\dot{x} &= Ax + B_w w,\ x(0) = x_0,\\
y &= Cx + Dw,\\
z &= Lx.
\end{aligned}
$$

Unknown Filter Dynamics

$$
\begin{aligned}
\dot{x}_F &= A_F x_F + B_F y,\ x_F(0) = x_{F_0},\\
\hat{z} &= C_F x_F.
\end{aligned}
$$

Augmented System Dynamics With

$$
x_e = \begin{bmatrix} x \\ x_F \end{bmatrix},\ \tilde{z} = z - \hat{z},
$$

dynamics is

$$
\begin{aligned}
\dot{x}_e &= \tilde{A}x_e + \tilde{B}w,\ x_e(0) = x_{e0}\\
\tilde{z} &= \tilde{C}x_e.
\end{aligned}
$$

$\mathcal{H}_2$ Filtering

Problem Formulation (contd.)

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix}, \ \tilde{z} = z - \hat{z},$$

dynamics is

$$\begin{aligned} \dot{x}_e &= \tilde{A}x_e + \tilde{B}w, \ x_e(0) = x_{e_0} \\ \tilde{z} &= \tilde{C}x_e + \tilde{D}w, \end{aligned}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} L & -C_F \end{bmatrix}.$$

$\mathcal{H}_2$ Filtering

Synthesis

Optimization Problem

$$\min_{A_F, B_F, C_F} \gamma, \; \|\hat{G}_{w\to\tilde{z}}\|_2 < \gamma.$$

$\mathcal{H}_2$ Filtering

Synthesis

Optimization Problem

$$\min_{R,X,M,N,Z,Q} \gamma$$

subject to

$$X > 0, \quad R - X > 0, \quad \mathbf{tr}\, Q < \gamma^2,$$

$$\begin{bmatrix} -Q & * & * \\ L^T & -R & * \\ -N^T & -X & -X \end{bmatrix} < 0,$$

$$\begin{bmatrix} RA + A^T R + ZC + C^T Z^T & * & * \\ M^T + ZC + XA & M^T + M & * \\ B^T R + D^T Z^T & B^T X + D^T Z^T & -I \end{bmatrix} < 0.$$

Proof:

Advances in Linear Matrix Inequality Methods in Control – edited by Laurent El Ghaoui, Silviu-Iulian Niculescu