AERO 422: Active Controls for Aerospace Vehicles

Dynamic Response

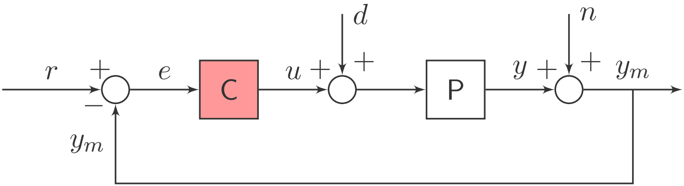
Raktim Bhattacharya

Laboratory For Uncertainty Quantification

Aerospace Engineering, Texas A&M University.

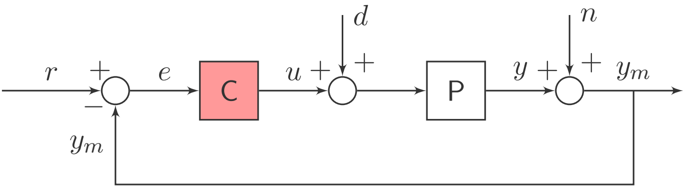
Previous Class

- Laplace transforms
- Transfer functions – from ordinary linear differential equations
- System interconnections
- Block diagram algebra – simplification of interconnections
- General feedback control system interconnection.



Transfer Functions

Standard Control System



Compactly

$$\begin{pmatrix} E(s) \\ Y(s) \end{pmatrix} \longleftarrow \boxed{\begin{array}{c|c|c} G_{er}(s) & G_{ed}(s) & G_{en}(s) \\ \hline G_{yr}(s) & G_{yd}(s) & G_{yn}(s) \end{array}} \longleftarrow \begin{pmatrix} R(s) \\ D(s) \\ N(s) \end{pmatrix}$$

Or

$$\begin{pmatrix} E(s) \\ Y(s) \end{pmatrix} = \left[\begin{array}{c|c|c} G_{er}(s) & G_{ed}(s) & G_{en}(s) \\ \hline G_{yr}(s) & G_{yd}(s) & G_{yn}(s) \end{array}\right] \begin{pmatrix} R(s) \\ D(s) \\ N(s) \end{pmatrix}$$

Response to Input

$$\begin{pmatrix} E \\ Y \end{pmatrix} = \left[\begin{array}{c|c|c} G_{er} & G_{ed} & G_{en} \\ \hline G_{yr} & G_{yd} & G_{yn} \end{array}\right] \begin{pmatrix} R \\ D \\ N \end{pmatrix}$$

implies

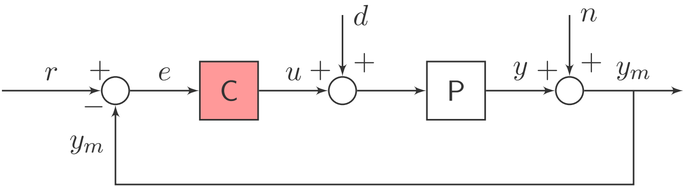
$$\begin{aligned} E &= G_{er}R + G_{ed}D + G_{en}N, \\ Y &= G_{yr}R + G_{yd}D + G_{yn}N. \end{aligned}$$

Therefore,

$$\begin{aligned} e(t) &= \mathcal{L}^{-1}\left\{G_{er}R\right\} + \mathcal{L}^{-1}\left\{G_{ed}D\right\} + \mathcal{L}^{-1}\left\{G_{en}N\right\}, \\ y(t) &= \mathcal{L}^{-1}\left\{G_{yr}R\right\} + \mathcal{L}^{-1}\left\{G_{yd}D\right\} + \mathcal{L}^{-1}\left\{G_{yn}N\right\}. \end{aligned}$$

Given signals $r(t), d(t), n(t)$, we can determine $e(t)$ and $y(t)$.

Definition of Various Transfer Functions



- Derive G_{er}.
- Ignore d and n.

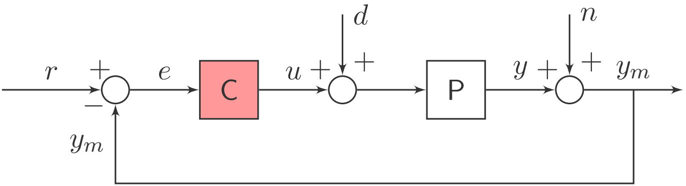
$$E = R - Y_m, \qquad U = C(s)E,$$
$$Y = P(s)(U + D) = P(s)U, \qquad Y_m = Y + N = Y.$$

- Simplification

$$\frac{E}{R} = G_{er} = \frac{1}{1 + PC}.$$

Definition of Various Transfer Functions

contd.



$$G_{er} = \frac{1}{1+PC}, \quad G_{ed} = -\frac{P}{1+PC}, \quad G_{en} = -\frac{1}{1+PC},$$
$$G_{yr} = \frac{PC}{1+PC}, \quad G_{yd} = \frac{P}{1+PC}, \quad G_{yn} = -\frac{PC}{1+PC}.$$

- Learn to derive these expressions.
- Denominator of all transfer functions: $1 + PC$.

Example

Let

$$P = \frac{1}{(s+1)(s+2)}, C = 1.$$

Look at

$$G_{yr} = \frac{PC}{1+PC} = \frac{\frac{1}{(s+1)(s+2)}}{1+\frac{1}{(s+1)(s+2)}} = \frac{1}{1+(s+1)(s+2)}$$

Response to reference $r(t) = 1(t)$?

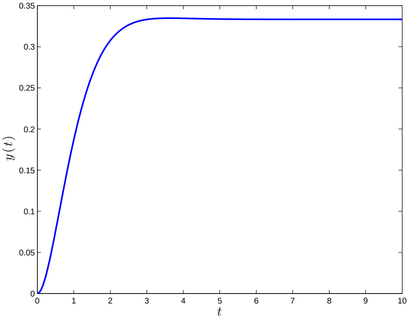
$$Y(s) = G_{yr}(s)R(s) = \frac{1}{1+(s+1)(s+2)}\mathcal{L}\{1(t)\}$$

$$= \frac{1}{1+(s+1)(s+2)} \cdot \frac{1}{s} = \frac{1}{s(s^2+3s+3)}.$$

$$\Rightarrow y(t) = \frac{1}{3} - \frac{e^{-\frac{3t}{2}}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right) + \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3}$$

Example

Response to $r(t) = 1(t)$.



$$y(t) = \frac{1}{3} - \frac{e^{-t}\left(\cos\left(\sqrt{2}\,t\right) + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2}\right)}{3}$$

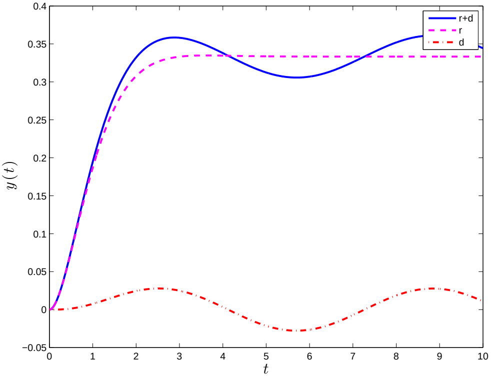
Example

What about d(t) = $\sin(t)/10$?

$$Y(s) = G_{yd}(s)D(s) = \frac{P}{1+PC}\mathcal{L}\left\{\sin(t)/10\right\}$$

$$y(t) = \frac{\sin(t)}{65} - \frac{3\,\cos(t)}{130} + \frac{3\,\mathrm{e}^{-\frac{3\,t}{2}}\left(\cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{5\,\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{9}\right)}{130}.$$

Total Response



$$y(t) = \mathcal{L}^{-1}\left\{G_{yr}R\right\} + \mathcal{L}^{-1}\left\{G_{yd}D\right\}.$$

In general $d(t)$ and $n(t)$ are more complicated functions of time.

Poles, Zeros & Causality

Poles and Zeros

- Given transfer function $G(s)$ between two signals
- Let $G(s) := \frac{N_G(s)}{D_G(s)}$ Rational polynomials
- Roots of $N_G(s)$ are called zeros of $G(s)$
  - Let there be m roots of $N_G(s)$
  - $N_G(s) = \Pi_{i=1}^m (s - z_i)$
- Roots of $D_G(s)$ are called poles of $G(s)$.
  - Let there be n roots of $D_G(s)$
  - $D_G(s) = \Pi_{i=1}^n (s - p_i)$
- The equation $D_G(s) = 0$ is called the characteristic equation
- $G(s)$ often is written as

$$G(s) = \frac{\Pi_{i=1}^m (s - z_i)}{\Pi_{i=1}^n (s - p_i)}$$

- Relative degree: $n - m$
- $n > m$ $G(s)$ is strictly proper
- $n \geq m$ $G(s)$ is proper

Causality

Causal

- A system is causal when the effect does not anticipate the cause; or zero input produces zero output
- Its output and internal states only depend on current and previous input values
- Physical systems are causal

Causality

contd.

Acausal

- A system whose output is nonzero when the past and present input signal is zero is said to be anticipative
- A system whose state and output depend also on input values from the future, besides the past or current input values, is called acausal
- Acausal systems can only exist as digital filters (digital signal processing).

Causality

contd.

Anti-Causal

- A system whose output depends only on future input values is anti-causal
- Derivative of a signal is anti-causal.

Causality

contd.

- Zeros are anticipative
- Poles are causal
- Overall behavior depends on m and n.

- Causal: $n > m$, strictly proper
- Causal: $n = m$, still causal, but there is instantaneous transfer of information from input to output
- Acausal: $n < m$

Example

- System $G_1(s) = s$
- Input $u(t) = \sin(\omega t)$, $U(s) = \frac{\omega}{s^2+\omega^2}$
- $y_1(t) = \mathcal{L}^{-1}\left\{G_1(s)U(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s\omega}{s^2+\omega^2}\right\} = \omega\cos(\omega t)$, or

$$\begin{aligned} u(t) &= \sin(\omega t) \\ y_1(t) &= \omega \sin(\omega t + \pi/2) \\ &= \omega u\left(t + \frac{\pi}{2\omega}\right) \text{ output leads input, anticipatory} \end{aligned}$$

Example

contd.

- System $G_2(s) = \frac{1}{s}$
- Input $u(t) = \sin(\omega t)$, $U(s) = \frac{\omega}{s^2+\omega^2}$
- $y_2(t) = \mathcal{L}^{-1}\{G_2(s)U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{\omega}{s^2+\omega^2}\right\} = \frac{1}{\omega} - \frac{\cos(\omega t)}{\omega}$, or

$$
\begin{aligned}
u(t) &= \sin(\omega t) \\
y_2(t) &= \frac{1}{\omega} + \frac{\sin(\omega t - \pi/2)}{\omega} \\
&= \frac{1}{\omega} + \frac{u(t - \frac{\pi}{2\omega})}{\omega} \quad \text{output lags input, causal}
\end{aligned}
$$

Time Response

Final Value Theorem – DC Gain

Given transfer function $G(s)$, DC gain is defined by

$$\text{DC Gain} = \lim_{s\to 0} G(s)$$

- Steady-state output of $G(s)$ to a step
- Only applicable to systems with poles in LHP, or stable systems Final value is bounded
- Steady state gain $(\lim_{t\to\infty})$ response

What happens for causal and acausal systems?

Initial Value Theorem - Transients

Given transfer function $G(s)$, transient response is given

$$y(0^+) = \lim_{s\to\infty} sG(s)$$

Example Let $G(s) = \frac{3}{s(s-2)}$, unstable system. Impulse response

$$y(0^+) = \lim_{s\to\infty} sG(s) = \lim_{s\to\infty} s\frac{3}{s(s-2)} = 0.$$

What happens for causal and acausal systems?

Impulse Response

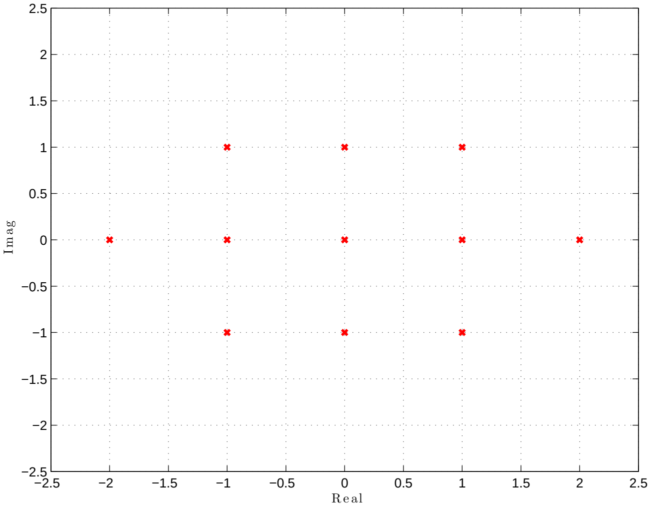
- Let $G(s)$ be given transfer function
- Let $u(t) = \delta(t)$, impulse function
- $U(s) = \mathcal{L}\{\delta(t)\} = 1$
- $Y(s) = G(s)U(s) = G(s) \cdot 1 = G(s)$
- $y(t) = \mathcal{L}^{-1}\{G(s)\}$ is the natural response of $G(s)$

Impulse response is used to obtain transfer function of a system from experimental data.

- Excite a system with $\delta(t)$ True $\delta(t)$ is difficult to realize in real world
- Record $y(t)$ from sensor data
- $\mathcal{L}\{y(t)\}$ provides $G(s)$

System Response and Pole Locations

Concept of Stability



System Response and Pole Locations

contd.

- Each pole (real, complex pair) represents a mode of the response
- Total response is addition of all the modes
- If any one mode is divergent/unstable, the total response is divergent/unstable
- For a mode $\sigma \pm j\omega_d$
  - $\sigma < 0 \Rightarrow$ convergent/stable
  - ω_d damped frequency
  - $\omega_n := \sqrt{\sigma^2 + \omega_d^2}$: natural frequency
  - $\zeta := \frac{\sigma}{\omega_n}$: damping ratio

Example

$$G(s) = \frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

Impulse response: $y(t) = Ae^{-at} + Be^{-bt}$

System Response and Zero Locations

- Let $G(s) = (s+a)G_0(s)$, where $G_0(s)$ has no zeros
- Response of $G_0(s)$ to $u(t)$ is

$$Y_0(s) = G_0(s)U(s)$$

- Response of $G(s)$ to $u(t)$ is

$$\begin{aligned} Y(s) &= (s+a)G_0(s)U(s) \\ &= sG_0(s)U(s) + aG_0(s)U(s) \\ &= sY_0(s) + aY_0(s) \end{aligned}$$

Zeroes adds signal derivative

$$y(t) = \frac{dy_0(t)}{dt} + ay_0(t)$$

System Response and Zero Locations

Effect of zero near a pole

Let system be

$$G(s) = \frac{s + (a + \epsilon)}{(s + a)(s + b)} = \frac{\epsilon}{b - a}\frac{1}{s + a} + \frac{b - (a + \epsilon)}{b - a}\frac{1}{s + b}$$

What happens when $\epsilon \to 0$?

System Response and Zero Locations

A zero near the origin

Case 1

- $G(s) = (s+z)G_0(s)$
- DC Gain of $G(s)$ is

$$\lim_{s\to 0} G(s) = \lim_{s\to 0} sG_0(s) + z\lim_{s\to 0} G_0(s) = z\lim_{s\to 0} G_0(s)$$

Case 2

- $G(s) = (s/z+1)G_0(s)$
- DC gain of $G(s)$ is

$$\lim_{s\to 0} G(s) = \frac{1}{z}\lim_{s\to 0} sG_0(s) + \lim_{s\to 0} G_0(s) = \lim_{s\to 0} G_0(s)$$

Preferable to keep DC gain unaffected.

System Response and Zero Locations

A zero near the origin (contd.)

- $G(s) = (s/z + 1)G_0(s)$
- Let $Y_0(s) = G_0(s)U(s)$ be response to input $U(s)$
- Response of $G(s)$ is

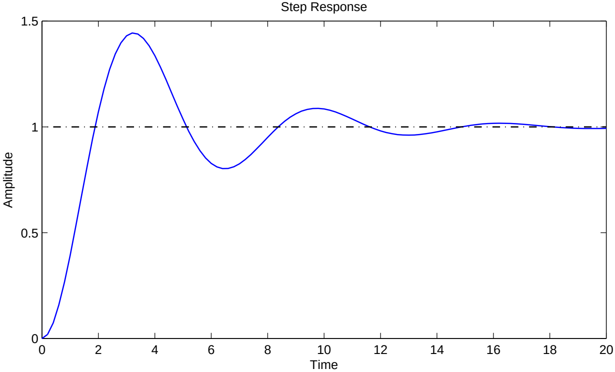
$$
\begin{aligned}
Y(s) &= (s/z + 1)G_0(s)U(s) = \frac{1}{z}sG_0(s)U(s) + G_0(s)U(s) \\
&= \frac{1}{z}sY_0(s) + Y_0(s)
\end{aligned}
$$

A zero near origin significantly amplifies the derivative of the response

$$
y(t) = \frac{1}{z}\frac{dy_0(t)}{dt} + y_0(t)
$$

Step Response

Time Domain Performance Specification



Second Order System: poles $= \sigma \pm j\omega_d$, $\omega_n = \sqrt{\sigma^2 + \omega_d^2}, \zeta = \sigma/\omega_n$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_r = \frac{1.8}{\omega_n} \qquad t_s = \frac{4.6}{\sigma}$$

Step Response

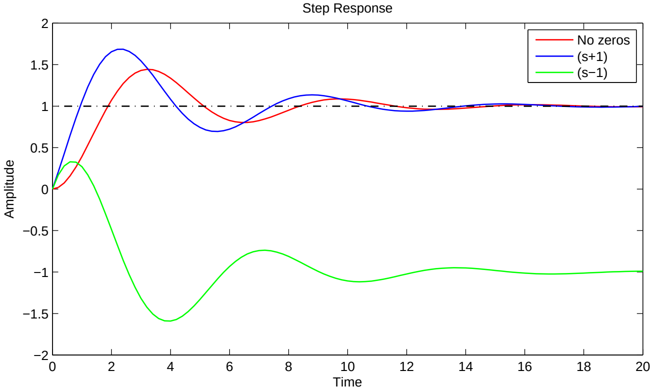
Time Domain Performance Specification – Second Order Systems

Desired Location of Poles

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_r = \frac{1.8}{\omega_n} \qquad t_s = \frac{4.6}{\sigma}$$

$$\omega_n \geq 1.8/t_r \qquad \zeta \geq \zeta(M_p) \qquad \sigma \geq 4.6/t_s$$

Step Response with Zeros



$$y(t) = \frac{dy_0(t)}{dt} + ay_0(t)$$

Stability Analysis

Various Notions of Stability

Basic Idea

- Disturbances/perturbations $\to 0$ as $t \to \infty$
- Refinements based on how they go to zero
- We talk about stability of the origin

Various Notions of Stability

contd.

- The origin is usually the equilibrium or trim point of the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

- Recall $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ are trim points, i.e.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = 0$$

- Here we study the stability of the perturbation dynamics

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\tilde{\boldsymbol{u}}, \quad \boldsymbol{A} := \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}, \quad \boldsymbol{B} := \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\Big|_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})},$$

where $\boldsymbol{x} = \tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}$ and $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \bar{\boldsymbol{u}}$.

Various Notions of Stability

contd.

- Stability analysis is concerned with behavior of $\lim_{t\to\infty} \boldsymbol{x}(t)$
- Equivalently study of $\lim_{t\to\infty} \tilde{\boldsymbol{x}}(t)$, for some $\tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0$,

$$\lim_{t\to\infty} \tilde{\boldsymbol{x}}(t) \to 0 \Leftrightarrow \lim_{t\to\infty} \boldsymbol{x}(t) \to \bar{\boldsymbol{x}}$$

- We study 3 kinds of stability
  1. Lyapunov stability
  2. Asymptotic stability
  3. Exponential stability

Lyapunov Stability

Aleksandr Mikhailovich Lyapunov

(1857–1918)

(Image: Wikipedia)

If for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, if

$$\|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| < \delta$$

then $\forall t \geq 0$ we have

$$\|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}\| < \epsilon.$$

How is it related to the poles of the system?

Asymptotic Stability

The equilibrium point is said to be asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ such that if

$$\|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| < \delta,$$

then

$$\lim_{t\to\infty} \|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}\| = 0.$$

How is it related to the poles of the system?

Exponential Stability

The equilibrium point is said to be exponentially stable if it is asymptotically stable and if there exists $\alpha, \beta, \delta > 0$ such that if

$$\|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| < \delta,$$

then

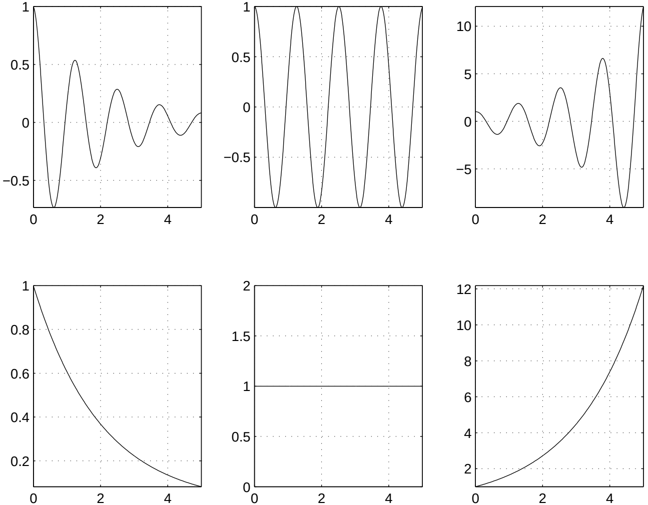
$$\|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}\| \leq \alpha \|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| e^{-\beta t}, \text{ for } t \geq 0.$$

- ES $\implies$ AS $\implies$ LS not the other way around
- β is called the Lyapunov exponent

How is it related to the poles of the system?

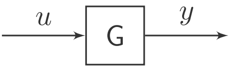
Stability of Linear Systems

Depends on location of poles



Input Output Stability

Bounded Input Bounded Output



- Given $|u(t)| \leq u_{\max} < \infty$, what can we say about $\max_t |y(t)|$?
- Recall

$$Y(s) = G(s)U(s) \implies y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau.$$

Therefore,

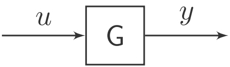
$$|y(t)| = \left|\int hud\tau\right| \leq \int |h||u|d\tau \leq u_{\max}\int |h(\tau)|d\tau. \text{ Cauchy-Schwarz}$$

Bound on output $y(t)$

$$\max_t |y(t)| \leq u_{\max}\int |h(\tau)|d\tau$$

Input Output Stability

Bounded Input Bounded Output



$$\max_t |y(t)| \leq u_{\max} \int |h(\tau)| d\tau$$

BIBO Stability

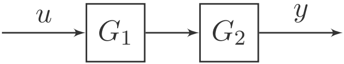
If and only if

$$\int |h(\tau)| d\tau < \infty.$$

(LTI): **Re** $p_i < 0 \implies$ BIBO stability

BIBO Stability

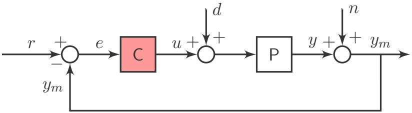
Interconnected Systems



- Given G_1 and G_2 are BIBO stable, is the above interconnection BIBO stable?

Input Output Stability

Pole Zero Cancellations



- Let

$$C(s) = \frac{s-1}{s+1}, \quad P(s) = \frac{1}{s^2-1} \quad \text{Pole Zero Cancellation}$$

- Look at transfer functions

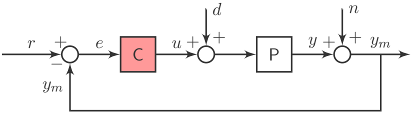
$$G_{yr} = \frac{PC}{1+PC} = \frac{1}{s^2+2s+2} \quad \text{poles:} -1 \pm i$$

$$\text{Unstable } G_{yd} = \frac{P}{1+PC} = \frac{s+1}{s^3+s^2-2} \quad \text{poles:} -2, 1$$

Input/output stability $\not\Rightarrow$ MIMO system stability (internal stability).

Input Output Stability

Pole Zero Cancellations

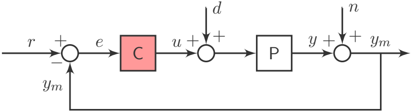


- Checking all TFs is tedious

$$G_{er} = \frac{1}{1+PC}, \quad G_{ed} = -\frac{P}{1+PC}, \quad G_{en} = -\frac{1}{1+PC},$$
$$G_{yr} = \frac{PC}{1+PC}, \quad G_{yd} = \frac{P}{1+PC}, \quad G_{yn} = -\frac{PC}{1+PC}.$$

- Just check zeros of $1 + PC$ No pole-zero cancellations

Internal Stability



Theorem

The above MIMO system is internally stable **iff**

1. The transfer function $1 + PC$ has no zeros in **Re** $s \geq 0$
2. There is no pole-zero cancellation in **Re** $s \geq 0$ when the product PC is formed

Internal stability ensures internal signals are not unbounded.